Demand, cost and information linkages in markets with decentralized information

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Abstract

How do linkages between markets affect the way producers competing in supply schedules use dispersed information about their costs? On their own, both the demand and the cost linkage let the error in one market's public information distort output in the other, and raise welfare losses. A demand linkage acts only through prices: it raises the efficient weight on private information about the own cost, toward the market's excessive weight, and lowers the share of the value of cost information that is lost. A cost linkage also acts through marginal costs: it lowers this efficient weight and raises this share. The two effects cancel when demand and cost parameters are proportional: what matters is the misalignment of the two linkages, not their strength. In symmetric markets, correlated public information moves the response to the other market against both structural linkages and can offset either of them. When firms learn from prices, the demand and the cost linkage also change what the prices reveal and make them correlated public signals. The efficient welfare loss still falls as information becomes more precise, but the weights on private information about the own cost can move in either direction as either linkage strengthens.

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1 Introduction

Markets are rarely isolated from each other. Two goods can be substitutes or complements for consumers. They can be produced with shared inputs or capacity, so that producing more of one raises or lowers the cost of producing the other. And what producers know about the costs of the two goods often comes from related sources, so that the errors of their estimates are correlated. Such linkages arise, for instance, between electricity price zones connected by transmission lines, or between two commodities processed from the same input: markets in which producers compete in supply schedules and hold dispersed information about common cost shocks. How does each of these linkages affect the way producers use their information, and how much welfare is lost as a result?

I answer this question in the framework of Vives (2017; V17 henceforth), who studies a single market. A continuum of firms face a common unknown cost parameter, receive private and public signals about it, and submit supply schedules. Firms rely on their private information more than is efficient, because they do not take into account how their responses to it move the price. The benchmark for efficiency is the team solution: the use of information that maximizes expected total surplus subject to the schedule of each firm depending only on its own signals. I extend this model to two markets. Each firm produces both goods and receives a private and a public signal about each cost, and the markets are linked in three ways: through demand, when the goods are substitutes or complements in consumption; through costs, when they are substitutes or complements in production; and through information, when the errors of the public estimates of the two costs are correlated.

What the demand and the cost linkage do alike. Switched on alone, each structural linkage carries the error of the public estimate of one cost into the output of the other good. Under a demand linkage, an error in the public estimate of the cost of good 2 distorts the output of good 2 and hence the price of good 1; under a cost linkage, it enters the output of good 1 directly, through the response of the firm to its estimate of the cost of good 2.

The team solution offsets this error with private information about the other market, which reveals the error on average across firms, and I call this response cross-market hedging. Under both linkages the hedge has the sign of the linkage and grows with its strength, and the responses of the team to private information and its welfare loss depend on the two linkages only through a single common index (Proposition 4). Both linkages also raise welfare losses in absolute terms, for a common reason. With substitutes, shifting output from one good to the other barely moves prices (under a demand linkage) or marginal costs (under a cost linkage), because the effect of the rise in one output is offset by the fall in the other; with complements, the same is true of expanding or contracting both outputs together. Outputs therefore respond strongly to the estimates of the relative cost of the two goods, with substitutes, or of their common cost level, with complements, so errors in these estimates turn into large output errors.

Where they differ. Switched on alone, the two linkages have opposite effects on the weight that firms should put on private information about their own cost, which compares their response to this information with their response to the cost itself under full information, and on the share of the value of cost information that is lost (Theorem 1). A stronger demand linkage raises the team-efficient weight, toward the weight used in the market, and lowers the share lost by both solutions. A stronger cost linkage lowers the team-efficient weight, away from the market weight, and raises the share lost. The difference comes from a second channel that only the cost linkage has: it also ties the two goods together inside each firm, through its marginal costs. Even under full information, each output then responds more strongly to its own cost at given prices. The market, which responds to private information as if prices did not react to output, scales up its response one for one with this full-information response. The response of the team grows only through the shared channel, as it would under a demand linkage, and as a share of the full-information response it falls. Under a demand linkage there is no second channel, so the response of the team grows while that of the market stays the same, and the gap between them narrows. The same distinction

governs welfare. In the market, the demand linkage raises the aggregate losses and the value of cost information in the same proportion and leaves the losses from the dispersion of outputs across firms unchanged, so the share lost falls; the cost linkage raises the dispersion losses faster than the value of information, so the share lost rises, and the share lost by the team moves in the same direction. For the same reason the demand linkage shrinks the set of parameters for which more precise private information increases the welfare loss of the market, and the cost linkage enlarges it (Proposition 5).

The information linkage. Correlated public information does not change payoffs, only information. When the errors of the public estimates of the two costs are correlated, private information about the other cost also reveals part of the error in the public estimate of the own cost, and correcting for this error calls for a response to the other market of the sign opposite to the correlation. For symmetric markets, the information linkage therefore pushes the response to the other market in the direction opposite to both structural linkages, and I characterize in closed form where it exactly offsets each of them (Proposition 7). The demand linkage that offsets a given correlation depends only on market structure, because both of them act only through the public errors, while the cost linkage that offsets it also depends on the precisions of information, because it acts through the private signals as well. The main weight and the welfare loss of the team are concave in the correlation and need not be monotone in it (Proposition 6): with symmetric markets every error splits into a common and a relative component, and correlation makes public information about one of them better and about the other worse.

Aligned linkages. Since the demand and the cost linkage move the weight on own private information in opposite directions, their effects can cancel. With uncorrelated public signals, they cancel for every degree of interconnection when the demand matrix is proportional to the cost matrix: then interconnection, however strong, affects neither the team-efficient split between private and public information, nor the gap between the market and the team, nor the share of the value of cost information that is lost (Theorem 2). With

aligned linkages, prices and marginal costs resist output changes in the same directions and in the same proportions, so interconnection changes the scale of the problem but not its shape. What matters for the use of dispersed information is therefore the misalignment between the demand and the cost linkage, and not their strength.

Learning from prices. The results above are for firms that take the public signals as given. When their public information comes from prices instead (Section 7), the three linkages are no longer separate. Each price reflects both costs, and the structural linkages generate a correlation between the information contained in the two prices, i.e. an information linkage. The weights then depend on the linkages in ways that have no sign: the team weights and the weight of the market on private information about the own cost can be non-monotone in the demand and the cost linkage, and neither quasi-concave nor quasi-convex (Facts 1 and 2). What can still be proven concerns information rather than the linkages: the team loss is decreasing in the precision of every source of uncertainty (Proposition 9).

Relative to V17, Propositions 1 and 2, together with the comparative statics in the precisions of information of Appendix B.3, extend its single-market results to interconnected markets. All results on the effects of the linkages are new, since none of them has an analogue with a single good. Related work links markets or assets in one of these ways at a time: through correlated fundamentals in multi-asset markets with dispersed information, through demand in the analysis of information sharing in oligopoly, and through costs in multimarket oligopoly under complete information; Subsection 1.1 discusses these and other related work.

1.1 Related literature

V17 is the baseline for this paper. It introduces a model of schedule competition in a single-good market with partially decentralized information in which agents condition on prices in the rational expectations tradition, and its focus is on the information externalities within that framework: their sign, and the difference between naive and fully rational agents. This paper focuses on the linkages between markets: what each of them does to the use of

information and to welfare, and why.

Information externalities have been studied in a wide range of models, mostly with a single good. This paper studies them in markets for several goods that are substitutes or complements in consumption or in production. To the best of my knowledge, the effects of linkages in consumption and in production on the use of dispersed information have not been studied in this literature.

The paper continues the strand of literature dedicated to the welfare analysis of economies with private or decentralized information (Cooper and John, 1988; Palfrey, 1985; Vives, 1984, 1988; Angeletos and Pavan, 2007, 2009). It extends the analysis to multiple interconnected markets and endogenous public information when the public signal is the price, which is influenced by the actions of agents. More specifically, to perform this analysis, it takes the model of strategic competition in schedules such as in Kyle (1985, 1989) and Vives (2011) but uses a continuum-of-agents framework. The idea of using the team solution as a welfare benchmark is not new and has been used in a wide strand of literature (Radner, 1979; Vives, 1988; Angeletos and Pavan, 2007). This idea follows Hayek (1945), in which private signals of agents cannot be aggregated via some social planner.

The results for naive firms confirm the usual intuition of informational externalities models (Vives, 1997; Amador and Weill, 2010, 2012) in a market game. It is worth noting that the information externalities of this paper are usually related to the inefficiency in competitive but incomplete markets and/or to the existence of decentralized information, because in that case, one of the main conditions of the first fundamental welfare theorem is violated. And hence market equilibria or competitive noisy rational expectations equilibria (REE), in which traders take into account information from prices, are definitely not constrained efficient in those circumstances (as in Laffont, 1985; Greenwald and Stiglitz, 1986).

Settings with several goods or assets and dispersed information do appear in a few strands of the literature, but there the goods are linked in one way only. In the literature on multi-asset financial markets the assets are linked through their fundamentals: their payoffs

are correlated, and traders learn about one asset from the prices of the others (Admati, 1985; Cespa and Foucault, 2014; Chabakauri et al., 2022). This is the counterpart of the information linkage of this paper, which also captures correlated fundamentals, while the demand and the cost linkage are absent. Rostek and Yoon (2021, 2024) study trading in demand schedules for several assets with complementarities and substitutabilities in their payoffs and show how the design of market clearing, which determines on which prices the demand for each asset can be contingent, affects price impact and welfare; there is no production side in these models, and in Rostek and Yoon (2021) the private information of traders concerns their endowments rather than the asset values.

In industrial organization, products that are substitutes or complements in demand are standard in the analysis of information sharing in oligopoly (Vives, 1984; Raith, 1996), where the question is whether firms have incentives to share their private information, and Bulow et al. (1985) show, under complete information, how the actions of a firm in one market affect its competitors in another when the two markets are linked through its costs. Interconnected markets are also central in the analysis of wholesale electricity markets, where producers compete in supply functions and neighboring zones are linked by a transmission network (Wilson, 2008; Holmberg and Philpott, 2018); this literature, however, studies market power under symmetric information rather than the use of dispersed private information. In macroeconomics, dispersed information has been studied in economies where goods are linked through demand (Angeletos and La'O, 2020) or through input–output linkages (La'O and Tahbaz-Salehi, 2022), with a focus on monetary policy rather than on how producers use dispersed information about their costs. Finally, in network games with incomplete information (de Martí and Zenou, 2015) agents interact through a network with strategic complementarities whose strength is only partially known to them; the interaction is a primitive there, rather than the result of demand or cost linkages between markets. This paper puts the three kinds of linkage, through demand, through costs and through information, into one model of schedule competition.

2 The Model

This model lies within the general framework of linear-quadratic Gaussian models, applying it to the case of market schedule competition, and follows many common assumptions. The notation usually follows the baseline paper of V17, with some small deviations.

Let me introduce here in advance a common notation for the precision of a random variable y: $\tau_y = \sigma_y^{-2}$. Also, I allow $\tau_y = \infty$ for some variables, which is equivalent to having no noise in a signal (perfect precision).

2.1 Market structure

Consider a continuum of firms indexed within the interval $[0, 1]$ (endowed with the Lebesgue measure), where x_{i1}, x_{i2} is the two-dimensional output of firm i, produced at cost

$$C(x_{i1}, x_{i2}) = \theta_1 x_{i1} + \theta_2 x_{i2} + 0.5\lambda_1 x_{i1}^2 + 0.5\lambda_2 x_{i2}^2 + \lambda_3 x_{i1} x_{i2},$$

where parameters $\lambda_j > 0, j \in \{1, 2\}$ and λ_3 are known to the firms, but parameters $\theta_j \sim N(0, \tau_\theta^{-1}), j \in \{1, 2\}$ are unknown to firms (I assume 0 mean WLOG for simplicity), and they have information only about their prior. Here λ_3 is the measure of substitutability/complementarity between the two goods in production, and for the cost function to be strictly convex we need $\lambda_3^2 < \lambda_1 \lambda_2$.

Firms face two inverse demands for homogeneous products: $p_1 = \alpha_1 + u_1 - \beta_1 \tilde{x}_1 - \beta_3 \tilde{x}_2$, $p_2 = \alpha_2 + u_2 - \beta_2 \tilde{x}_2 - \beta_3 \tilde{x}_1$, where $\alpha_j, \beta_j > 0, j \in \{1, 2\}$ are known demand parameters, $u_j \sim N(0, \tau_u^{-1})$ is a random market shock, while $\tilde{x}_j = \int_0^1 x_{ij} di$ is the aggregate output.

Such demands can be derived from a quadratic utility function of a representative consumer with quasilinear preferences[1]. Here we can perceive β_3 as the measure of substitutability/complementarity between the two goods in consumption, and notice that for such

[1]This also restricts the model to have the same β_3 in both demands; otherwise, there is no feasible consumer surplus for the welfare maximization problem.

a representative consumer to have well-defined demands, we need $\beta_3^2 \leq \beta_1\beta_2$.

Throughout the paper I use the notation $\phi_j = \beta_j + \lambda_j$, $j \in \{1,2,3\}$, and $K_x = x_1x_2 - x_3^2$, $x \in \{\lambda, \beta, \phi\}$. It is also convenient to collect the cost and the demand parameters in the matrices $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_3 & \lambda_2 \end{pmatrix}$ and $B = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$, so that $\Phi = \Lambda + B$.

For some sections of this paper I additionally require homogeneous production and consumption characteristics, i.e. $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$, and this is specified beforehand.

2.2 Information structure

Each firm receives two private signals: one about each θ_j:

$$s_{ij} = \theta_j + \varepsilon_{ij}, j \in \{1,2\}, \text{ where } \varepsilon_{ij} \sim N(0, \tau_\epsilon^{-1}), \text{ are i.i.d. across firms.}$$

I also assume that $\int_0^1 \varepsilon_{ij} di = 0$ a.s., which is just the limit of the model with a finite number of firms, and hence the aggregation of all individual signals reveals the underlying uncertainty: $\int_0^1 s_{ij} di = \theta_j$. Also, to avoid multiple correlation sources, I require $\varepsilon_{i1} \perp \varepsilon_{i2}$.

As for public signals, I first specify the difference between the two subframeworks I consider.

Exogenous public information. In this case I require agents to be naive about the price that they see, similarly to the case of the fully cursed equilibrium of Eyster and Rabin (2005), suggesting that prices $p_j, j \in \{1,2\}$ have no information about parameters $\theta_j, j \in \{1,2\}$.

Then each firm receives two public signals:

$$z_j = \theta_j + \omega_j, j \in \{1,2\}, \omega_j \sim N(0, \tau_\omega^{-1}), \omega_j \perp u_k, \forall j,k \in \{1,2\}.$$

Here I define

$$\tau = \mathbb{V}(\theta_j - \mathbb{E}[\theta_j | z_1, z_2])^{-1}$$

as the precision of public information, which positively depends on τ_θ and on τ_ω, and

$$\rho = cor(\theta_1 - \mathbb{E}[\theta_1|z_1, z_2], \theta_2 - \mathbb{E}[\theta_2|z_1, z_2])$$

as the third linkage parameter of the model. It aggregates in one number whatever correlation there is in ω_j or in θ_j; notice that the latter also creates correlation in private signals, but this does not matter for the team solution, because private information shocks are not correlated with each other, while it still matters for the market equilibrium. In the special case $\theta_1 \perp \theta_2$ and $\omega_1 \perp \omega_2$ we have $\rho = 0$ and $\tau = \tau_\theta + \tau_\omega$, as derived in V17.

Endogenous public information. In this case I assume that firms are sophisticated and do take into account the information content of prices in the rational expectations sense, and this is the only source of the public signals about θ_j.

Then it can be shown that observing the vector of prices $p = (p_1, p_2)$ (this notation is used further for simplicity) is equivalent to observing public signals of the form $z_j = \theta_j + \omega_j, j \in \{1, 2\}$, where, however, $\omega_j \not\perp u_k$ and $\omega_1 \not\perp \omega_2$. The important feature of this case, and the reason why it is treated separately, in Section 7, is that here ρ is no longer a free parameter of the model: it is generated by β_3 and λ_3 themselves. For exact details, see Section 7.1.

Hence, in both cases each firm i would have an information set $I_i = \{s_{i1}, s_{i2}, z_1, z_2\}$, including all 4 signals; however, we need to take into account differences in the structure of public signals.

2.3 Three linkages

The model as it is written above contains exactly three parameters that couple the two markets, and it is worth listing them together, because the rest of the paper describes their effects:

(i) β_3, the *demand linkage*: goods are substitutes ($\beta_3 > 0$) or complements ($\beta_3 < 0$) in

consumption;

(ii) λ_3, the *cost linkage*: goods are substitutes ($\lambda_3 > 0$) or complements ($\lambda_3 < 0$) in production;

(iii) ρ, the *information linkage*: the public estimates of the two cost parameters have correlated errors.

If all three are set to zero, the two markets become perfectly independent, and each of them is exactly the economy studied in V17. Most results switch on one or two of them, but some hold with all three present: the independence of the market weights from β_3, λ_3 (Section 4.1), the team efficiency of Cournot competition (Proposition 1), all the results of Section 6 for symmetric markets, and, with endogenous public information, the joint parity of Section 7.3 and Proposition 9.

2.4 Market Game

The timing of events is as follows. At $t = 0$, the random variables $\theta_j, u_j, j \in \{1, 2\}$ are drawn but not observed. At $t = 1$, the aggregate consumer and firms form demand and supply plans. The consumer maximizes utility knowing the realization of u. Each firm submits supply schedules $X_{ij}(I_i; \cdot)$ contingent on its information set. The strategy of a firm is a map from the signal vector space to the space of schedules. At $t = 2$ the market clears, and the price is formed by finding p that equates demands and aggregate supplies $\int_0^1 X_{ij}(I_i; p)di$.

Let us assume that there is a unique price vector for any realization of the signals. Then, for a given profile of firms' schedules and realization of the signals, the profit for firm i is given by $\pi_i = p_1 x_{i1} + p_2 x_{i2} - C(x_{i1}, x_{i2})$, where p are realized prices, and $x_{ij} = X_{ij}(I_i; p)$ are realized outputs. This defines a Bayesian game in schedules, where firms maximize their expected profit conditional on observed signals: $\mathbb{E}[\pi_i | I_i]$.

The main interpretation of the model is competition in supply schedules in two homogeneous product markets, such as adjacent electricity price zones linked by a transmission

interconnector, or two commodity markets sharing a common input. Following Section 7 of V17, the model can also be inverted to describe trading in demand schedules for two assets with unknown values that are complements or substitutes in traders' preferences; I do not pursue this interpretation here.

I study linear Bayesian equilibria of the market schedules game in two versions: the exogenous and the endogenous public information cases. And in the next section I define the appropriate welfare benchmark for the use of information in that economy, which I also study in this paper.

3 Welfare benchmark

Consider an allocation assigning output x_{ij} to firm i with average output $\tilde{x}_j$. I use a utilitarian welfare criterion, which in this quasilinear world is equivalent to total surplus minus costs:

$$\text{TS} = \sum_{j=1}^{2}(\alpha_j + u_j - 0.5\beta_j\tilde{x}_j)\tilde{x}_j - \beta_3\tilde{x}_1\tilde{x}_2 - \int_0^1 \left[\sum_{j=1}^{2}\theta_j x_{ij} + 0.5\lambda_j x_{ij}^2 + \lambda_3 x_{i1}x_{i2}\right] di.$$

The first-best (full-information) allocation has all firms producing the same amount:

$$x_j^0 = \frac{\phi_{-j}(\alpha_j + u_j - \theta_j) - \phi_3(\alpha_{-j} + u_{-j} - \theta_{-j})}{K_\phi}, j \in \{1, 2\}.$$

As usual, here and below the index $-j$ means that we choose an index $k \in 1, 2, k \neq j$.

Denote by TS^0 the total surplus at the full-information first best. The first-best allocation is attained by the competitive market when there is full information (i.e. firms receive perfect signals about θ_j). The market equilibria I consider under asymmetric information do not attain the first best in general. The reason is that suppliers produce under uncertainty and rely on imperfect idiosyncratic estimation of the common cost component; hence, they end up producing different amounts even though costs are identical and strictly convex.

We have that for a symmetric allocation for which $\mathbb{E}[\int_0^1 (x_{ij} - \tilde{x}_j)^2 di] = \mathbb{E}[(x_{ij} - \tilde{x}_j)^2]$,

$\mathbb{E}[\text{TS}] = \mathbb{E}[\text{TS}^0] - \text{WL}$, where WL is the expected welfare loss:

$$\text{WL} = 0.5 \sum_{j=1}^{2} \lambda_j \mathbb{E}[(x_{ij} - \tilde{x}_j)^2] + \lambda_3 \mathbb{E}[(x_{i1} - \tilde{x}_1)(x_{i2} - \tilde{x}_2)] +$$

$$+0.5 \sum_{j=1}^{2} \phi_j \mathbb{E}[(\tilde{x}_j - x_j^0)^2] + \phi_3 \mathbb{E}[(\tilde{x}_1 - x_1^0)(\tilde{x}_2 - x_2^0)].$$

The first terms in WL correspond to distributive inefficiency (how distorted is the distribution of production of a given average quantity $\tilde{x}_j$). The last terms correspond to aggregate inefficiency (how distorted is the average quantity x_j while producing in a cost-minimizing way), which are proportional to the dispersion of the aggregate outputs $\mathbb{E}[(\tilde{x}_j - x_j^0)^2]$ and their covariance $\mathbb{E}[(\tilde{x}_1 - x_1^0)(\tilde{x}_2 - x_2^0)]$. Let me notice here that the two quadratic forms appearing in WL are exactly the ones generated by the matrices Λ and Φ, which is the reason why the effects of the demand and the cost linkage can be traced in the same object: they enter it through two matrices of the same shape.

To measure how well information is used, I compare the welfare loss with the value of information about costs,

$$\text{VI} = \frac{1}{2}\mathbb{E}[\theta' \Phi^{-1} \theta] = \frac{\text{tr}(\Phi^{-1})}{2\tau_\theta},$$

where the second equality holds for independent θ_1, θ_2, as whenever VI is used below. It is the welfare loss of the economy in which firms know only the prior about θ and do not learn from prices: they supply $\Lambda^{-1}p$, all firms produce the same output, and market clearing gives $\tilde{x} - x^0 = \Phi^{-1}\theta$. Since $\mathbb{E}[\text{TS}] - \mathbb{E}[\text{TS} \,|\, \text{prior only}] = \text{VI} - \text{WL}$, the relative loss $\frac{\text{WL}}{\text{VI}}$ is the share of the value of cost information that the economy fails to capture. I use VI rather than the first-best surplus $\mathbb{E}[\text{TS}^0]$, because the latter also contains the surplus from the average outputs, which is determined by the demand intercepts α_j and has nothing to do with information.

The welfare benchmark I use is the team solution. In it, the social planner maximizes expected total surplus constrained by the fact that the firm can condition its actions only on

its own observed information, i.e. subject to employing informationally decentralized strategies (as in Vives, 1988; Angeletos and Pavan, 2007). This team-efficient solution internalizes the payoff and information externalities of the actions of agents.

In the economy considered, if firms set quantities as strategies, conditioning only on their available signals as in a Cournot market, instead of using a supply function, i.e. without price conditioning, then the market solution would be team-efficient (see Vives, 1988, Section 4.6). That is, the Cournot private information equilibrium is team-efficient for given public information. This is not the case in the market game with price-contingent strategies, even when firms disregard the information content of the price (i.e. with exogenous public information), because of a pecuniary externality[2] in the use of private information. This replicates the analogous conclusion of V17 that the economy in which agents use non-price-contingent strategies displays exactly the right degree of coordination. In the present framework, however, it says something more than it does in a one-good model, and it is worth stating separately.

Proposition 1. *With non-price-contingent strategies the market equilibrium is team-efficient for any values of* β_3, λ_3 *and* ρ.

The equilibrium equations and the proof are given in Appendix B.1. In a paper in which nearly every object depends on the three linkage parameters, and in which much of Section 7 is devoted to the ways in which that dependence becomes intractable, Proposition 1 is the one statement that is completely insensitive to them. The informational inefficiency studied here is therefore a property of price-contingent competition and not a property of interconnection between markets: it is the conditioning on prices, and not the linkage, that creates the wedge.

[2]See the introduction of V17 about this externality

4 Characterization with exogenous public information

In this section, I consider the case in which firms are naive in the sense that they do not realize the informational value of the price, and I characterize the market equilibrium and the team solution.

4.1 Market equilibrium

At the market equilibrium, firm i solves

$$\max_{x_{i1},x_{i2}} \mathbb{E}[p_1 x_{i1} + p_2 x_{i2} - \theta_1 x_{i1} - \theta_2 x_{i2} - 0.5\lambda_1 x_{i1}^2 - 0.5\lambda_2 x_{i2}^2 - \lambda_3 x_{i1} x_{i2} | I_i].$$

The solution is both unique (given strict concavity of profits) and symmetric across firms (since the cost function and signal structure are symmetric across firms):

$$X_{ij}^M(I_i, p) = \frac{\lambda_{-j}(p_j - \mathbb{E}[\theta_j|I_i]) - \lambda_3(p_{-j} - \mathbb{E}[\theta_{-j}|I_i])}{K_\lambda}, j \in \{1, 2\}.$$

From the projection theorem for Gaussian variables we know that if $\rho = 0$, then $\mathbb{E}[\theta_j|I_i] = \gamma_j^M s_{ij} + (1 - \gamma_j^M)\hat{\theta}_j$, where $\hat{\theta}_j = \mathbb{E}[\theta_j|z_j]$ and $\gamma_j^M = \frac{\tau_\epsilon}{\tau_\epsilon + \tau}$ is the Bayesian weight. If $\rho \neq 0$, the estimator $\mathbb{E}[\theta_j|I_i]$ must use all 4 signals with non-zero weights and has no representation as a weighted average of two signals; its exact form is in the appendix. Averaging the supply schedules of firms and using the market-clearing conditions, one immediately sees that the equilibrium is unique, symmetric, and linear.

It is important to notice, and this fact is used repeatedly below, that *the weights that firms put on their signals in the market equilibrium depend only on the information parameters*, and not on β_3 or λ_3 at all. No change in demand parameters influences the optimal behavior of the firm in equilibrium. Consequently, every effect of the demand and the cost linkage that is described in this paper is an effect on the team solution and hence on the wedge between the two.

4.2 Team solution

At the team-efficient solution with exogenous public information, expected total surplus is maximized subject to firms using linear output strategies contingent only on the information set and prices, with no learning about θ from p.

The answer is pretty close to the market equilibrium with some important differences[3]. To shorten the formulas, let me denote $F_j = \phi_j \tau + \lambda_j \tau_\epsilon$, $j \in \{1,2,3\}$, and $K_F = F_1 F_2 - F_3^2$, in the same way as K_x in Section 2. Then, for $\rho = 0$,

$$X_{ij}^T(I_i, p) = \frac{\lambda_{-j}(p_j - \gamma_{jj}^T s_{ij} - (1-\gamma_{jj}^T)\hat{\theta}_j) - \lambda_3(p_{-j} - \gamma_{j(-j)}^T s_{i(-j)} - (1-\gamma_{j(-j)}^T)\hat{\theta}_{-j})}{K_\lambda},$$

$$\gamma_{jj}^T = \frac{K_\lambda F_{-j} \tau_\epsilon}{\lambda_{-j} K_F},\ j \in \{1,2\}, \qquad \gamma_{12}^T = \gamma_{21}^T = \frac{K_\lambda F_3 \tau_\epsilon}{\lambda_3 K_F}.$$

Here the γ's, as in the market equilibrium, are the weights on private information: how much firms rely on it or react to it. These weights minimize WL and optimally trade off the sources of inefficiency among decentralized strategies. It is also convenient to have these formulas in matrix form. If I write the team schedules as $X_i^T = \Lambda^{-1} p - C^T s_i - (\Lambda^{-1} - C^T)\hat{\theta}$, where $s_i = (s_{i1}, s_{i2})'$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)'$, then the formulas above say simply that

$$C^T = \tau_\epsilon(\tau\Phi + \tau_\epsilon\Lambda)^{-1}, \qquad \text{while} \qquad C^M = \gamma^M \Lambda^{-1} = \tau_\epsilon(\tau\Lambda + \tau_\epsilon\Lambda)^{-1}$$

for the market schedules of Section 4.1. So the market weights private signals exactly as the team would if the demand matrix B were zero, i.e. if prices did not react to aggregate output, and this is the pecuniary externality in its most compact form. It also shows that the assumption of equally precise information in the two markets is inessential: both problems separate by the information about each cost parameter, and the precisions of this information enter only the corresponding column.

Remark 1. With market-specific precisions $\tau_{\epsilon j}, \tau_j$ and $\rho = 0$, the j-th columns of C^T and C^M

[3]See Appendix A.1 for the exact derivation details

are those of $\tau_{\epsilon j}(\tau_j \Phi + \tau_{\epsilon j}\Lambda)^{-1}$ and $\tau_{\epsilon j}(\tau_j \Lambda + \tau_{\epsilon j}\Lambda)^{-1}$, so all results on the weights for $\rho = 0$ hold market by market (Appendix B.2).

4.2.1 Main weights

The main weights γ_{jj}^T are the weights on the private signal about the cost of the own good, and the first question is how they compare with the Bayesian weight γ^M used in the market. Although the linkages move γ_{jj}^T in either direction (Section 5), they never change the answer.

Proposition 2. *Let $0 < \tau_\epsilon < \infty$ and $\rho = 0$. Then for any β_3, λ_3, at the unique market equilibrium with exogenous public information, firms give relatively more weight to the private information than at the team-optimal solution:*

$$\gamma_j^M = \frac{\tau_\epsilon}{\tau_\epsilon + \tau} \geq \gamma_{jj}^T = \frac{K_\lambda F_{-j} \tau_\epsilon}{\lambda_{-j} K_F}, \; j \in \{1, 2\},$$

with strict inequality unless $\beta_3^2 = \beta_1 \beta_2$ and $\lambda_{-j}\beta_j = \lambda_3 \beta_3$.

For $\beta_3 = \lambda_3 = 0$ this is the central finding of V17, and the proposition says that no configuration of the structural linkages can reverse it: the linkages move the team-efficient weight, as Section 5 shows, but this weight never exceeds the Bayesian weight that firms use in the market. In other words, the pecuniary externality that makes firms overreact to their private information survives any degree of interconnection, and all the effects of the demand and the cost linkage described below are effects on the size of this wedge, not on its sign. The comparative statics of both weights and of the welfare losses with respect to the precisions of information, which were the other part of the analysis of V17, carry over for both solutions and for any β_3, λ_3 (Appendix B.3), with one exception: the effect of more precise private information on the market loss, which the linkages do change and which is the subject of Proposition 5.

Remark 2. To make the market solution efficient, we need to either

i) have perfect public information: $\tau = \infty$

ii) have totally uninformative or perfect private information: $\tau_\epsilon \in \{0, \infty\}$

iii) have the price independent of aggregate outputs and therefore of the cost shocks θ_j: $\beta_1, \beta_2, \beta_3 = 0$, so that there are no externalities whatsoever.

4.2.2 Side weights

The market reacts to the information about the other market only mechanically, through the cost linkage, while the team reacts to it for informational reasons as well. This reaction is described by the coefficient on the private signal $s_{i(-j)}$ in X_{ij}^T, which I call the side weight:

$$a^T = \frac{\lambda_3 \gamma_{12}^T}{K_\lambda} = \frac{F_3 \tau_\epsilon}{K_F}.$$

Let me notice that $F_3 = \beta_3 \tau + \lambda_3(\tau + \tau_\epsilon)$, so that the demand and the cost linkage enter the side weight with the same sign, which is worth keeping in mind for what follows.

With perfect public or private information the team schedule is the first-best one, $x_i^{\text{FB}} = \Lambda^{-1}(p - \theta)$, whose reaction to the other market is purely mechanical: its side weight is $\frac{\lambda_3}{K_\lambda}$, which is zero for cost-neutral goods. With imperfect information the side weight is non-zero even for cost-neutral goods, although the signals about the two markets are independent.

The side weight also depends on the precisions of information, and here the two structural linkages give it different shapes. When private information is useless the team puts no weight on it, and when it is perfect the team schedule is the first-best one. So under the demand linkage alone the side weight vanishes at both extremes, while under the cost linkage alone it moves from zero to the first-best side weight $\frac{\lambda_3}{K_\lambda}$. The next proposition describes how it moves between these extremes.

Proposition 3. *Let the public information be exogenous and $\rho = 0$.*

(i) If $\lambda_3 = 0$ and $\beta_3 \neq 0$, then the $|a^T|$'s are quasi-concave in τ_ϵ, τ, and they are maximized at $\frac{\tau}{\tau_\epsilon} = \sqrt{\frac{\lambda_1 \lambda_2}{\phi_1 \phi_2 - \beta_3^2}} < 1$.

(ii) If $\beta_3 = 0$ and $\lambda_3 \neq 0$, then the $|a^T|$'s are increasing in τ_ϵ and decreasing in τ, depending only on $\frac{\tau_\epsilon}{\tau}$, and they are smaller than the side coefficient of the market, $\frac{|\lambda_3|\gamma^M}{K_\lambda}$, which in turn is smaller than the first-best side weight $\frac{|\lambda_3|}{K_\lambda}$, to which $|a^T|$ converges as $\frac{\tau_\epsilon}{\tau} \to \infty$.

Part (ii) shows that under the cost linkage, unlike under the demand linkage, there is no interior optimum, because the side weight is dominated by the mechanical response to the other cost, which firms follow more closely as their private information becomes better. Measured against the market, the two linkages push the side weights in opposite directions: under the demand linkage the market ignores the private information about the other market while the team uses it, and under the cost linkage the team uses less of it than the market does. This is the first appearance of the contrast that Section 5 establishes for the main weights.

4.3 Cross-market hedging

To see why the team reacts to the information about the other market at all, let me write the terms of X_{ij}^T that concern the other market, apart from the price p_{-j}, as $\frac{\lambda_3}{K_\lambda}\hat{\theta}_{-j} + a^T(s_{i(-j)} - \hat{\theta}_{-j})$. The first term is the first-best response to the public estimate of the other cost, and the second one is a position in the difference between the private and the public signal about it, whose average over firms is exactly the public error $\theta_{-j} - \hat{\theta}_{-j}$. Since the markets are interconnected, this error reaches the output of good j even without a side weight: under the demand linkage it distorts the output of good $-j$ and hence the price p_j, and under the cost linkage it enters through the response to $\hat{\theta}_{-j}$. The side weight works against it, so that firms hedge the output of their good against the public error made in the other market. I refer to this mechanism as *cross-market hedging*, and it is the central economic force behind most of the results in this paper.

Hedging also explains the interior peak of the side weight under the demand linkage in Proposition 3(i). It requires two signals that are both informative but imperfect: if the private signal is nearly perfect ($\tau_\epsilon \to \infty$), firms hardly use the public one, so there is little

public error to hedge against, and if it is nearly worthless ($\tau_\epsilon \to 0$), it is useless as a hedge. The gains from hedging are therefore largest at intermediate relative precision, not at either extreme – the same logic that makes combining two noisy, non-redundant estimates most valuable when neither one dominates the other.

What the hedge does to prices depends on the linkage. In the original model of V17, if p^0 is the first-best price, then $\mathbb{E}[(p - p^0)^2] = \beta^2 \mathbb{E}[(\tilde{x} - x^0)^2]$, so that the noise in the price and the aggregate inefficiency are the same thing. With two markets this is no longer so. Now $p - p^0 = -B(\tilde{x} - x^0)$, so the noise in both prices, $\frac{1}{2}\mathbb{E}[(p - p^0)' B^{-1} (p - p^0)]$, is only the demand-side part of the aggregate losses,

$$0.5 \sum_{j=1}^{2} \beta_j \mathbb{E}[(\tilde{x}_j - x_j^0)^2] + \beta_3 \mathbb{E}[(\tilde{x}_1 - x_1^0)(\tilde{x}_2 - x_2^0)],$$

while the aggregate part as a whole, $\frac{1}{2}\mathbb{E}[(\tilde{x} - x^0)' \Phi (\tilde{x} - x^0)]$, also contains its cost side. Under the cost linkage alone the hedge makes prices less noisy as well: in my simulations the public error of the other market in each price is always smaller than under the best schedules without side weights. Under the demand linkage alone it is the other way round. The error $\theta_2 - \hat{\theta}_2$ moves the output of good 2, $\tilde{x}_2 \sim \ldots + \lambda_2^{-1}(1 - \gamma_2^T)(\theta_2 - \hat{\theta}_2)$, hence moves the price of good 1, $p_1 = \ldots - \beta_1 \tilde{x}_1 - \beta_3 \tilde{x}_2 \sim \ldots - \beta_3 \lambda_2^{-1}(1 - \gamma_2^T)(\theta_2 - \hat{\theta}_2)$, and hence, through the schedule of good 1, moves the output of good 1, $\tilde{x}_1 \sim \ldots - \beta_3 \lambda_1^{-1} \lambda_2^{-1}(1 - \gamma_2^T)(\theta_2 - \hat{\theta}_2)$. The side weight works against this movement, so it has the sign of β_3, and to first order in β_3 it is proportional to it, since γ_2^T also depends on β_3. But then $\beta_1 \tilde{x}_1$ moves with the error in the same direction as $\beta_3 \tilde{x}_2$, and p_1 becomes even noisier. So under the demand linkage the team stabilizes outputs at the expense of prices: given the main weights, the side weights lower the aggregate part of the losses at the cost of additional distributive ones, but they raise the public error of the other market in both prices.

5 The demand and the cost linkage

In this section the public signals are uncorrelated, $\rho = 0$, and I describe the effects of the demand and the cost linkage on the use of information and on welfare. The next theorem states them for each of the two linkages switched on alone.

Theorem 1 (Effects of the demand and the cost linkage)**.** *Let the public information be exogenous and* $\rho = 0$*.*

(i) If $\lambda_3 = 0$*, then the* γ^T*'s are even functions increasing in* $|\beta_3|$*, and the side weights* a^T*'s are odd functions increasing in* β_3*. Welfare losses* $\mathrm{WL}^T, \mathrm{WL}^M$ *are even functions increasing in* $|\beta_3|$*, but* $\frac{\mathrm{WL}^T}{\mathrm{VI}}, \frac{\mathrm{WL}^M}{\mathrm{VI}}$ *are even functions decreasing in* $|\beta_3|$*.*

(ii) If $\beta_3 = 0$*, then* $\gamma_{11}^T, \gamma_{22}^T$ *are even functions decreasing in* $|\lambda_3|$*, and the side weights* a^T*'s are odd functions increasing in* λ_3*. Welfare losses* $\mathrm{WL}^T, \mathrm{WL}^M$ *and* $\frac{\mathrm{WL}^T}{\mathrm{VI}}, \frac{\mathrm{WL}^M}{\mathrm{VI}}$ *are all even functions increasing in* $|\lambda_3|$*.*

Both parts are stated in terms of $|\beta_3|$ and $|\lambda_3|$ because the market equilibrium and the team solution are invariant under the joint sign change $(\beta_3, \lambda_3) \to (-\beta_3, -\lambda_3)$ together with the sign change of the coefficients on the signals about the other market (Appendix A.4.5), and with one of the linkages switched off this is just the sign change of the other one. When both linkages are present, only this joint parity survives, exactly as in the endogenous case (Section 7.3).

The theorem sorts the outcomes into two groups. On the side weights and on the absolute welfare losses the two linkages act alike: each of them raises the side weight in its own direction, and each raises both losses. On the main weights and on the losses relative to the value of information they act in opposite directions: the demand linkage raises the team-efficient weight on private information and makes both solutions relatively better, while the cost linkage lowers this weight and makes both of them relatively worse. The absolute losses by themselves say little about the use of information, since both linkages also change the

value of information, and I return to this point in Section 5.3. The two groups correspond to two channels: the linkages share one, which is responsible for everything they do alike (Section 5.1), and the cost linkage has one more, which is responsible for the differences (Section 5.2).

5.1 What both linkages do

Let me first write the team solution for each of the two linkages switched on alone.

Demand linkage alone ($\lambda_3 = 0$). The matrix Λ is diagonal, $F_3 = \beta_3\tau$, $K_F = F_1F_2 - \beta_3^2\tau^2$, and the team schedule of Section 4.2 can be written in a simpler form:

$$X_{ij}^T(I_i, p) = \lambda_j^{-1}(p_j - \gamma_j^T s_{ij} - (1-\gamma_j^T)\hat{\theta}_j) + a^T(s_{i(-j)} - \hat{\theta}_{-j}),\ j \in \{1,2\},$$

$$\gamma_j^T = \frac{\lambda_j F_{-j}\tau_\epsilon}{K_F}, \qquad a^T = \frac{\beta_3\tau\tau_\epsilon}{K_F}.$$

The formula has this shape for the same reason that the weights γ_j^T and $1-\gamma_j^T$ on the private and the public signal about the own cost sum to one: the two signals about each cost are two estimates of it, and their weights add up to the full-information response to this cost, so that the schedule is the first-best one when both signals are exact. For cost-neutral goods the full-information response to the other cost is zero, so the weights on $s_{i(-j)}$ and $\hat{\theta}_{-j}$ are a^T and $-a^T$, and the side weight is a pure position in $s_{i(-j)} - \hat{\theta}_{-j}$, whose average over firms is the public error in the other market, i.e. a pure hedge against this error in the sense of Section 4.3.

Cost linkage alone ($\beta_3 = 0$). Now $F_3 = \lambda_3(\tau + \tau_\epsilon)$, $K_F = F_1F_2 - \lambda_3^2(\tau+\tau_\epsilon)^2$, and the team schedule keeps the general form of Section 4.2:

$$X_{ij}^T(I_i, p) = \frac{\lambda_{-j}(p_j - \gamma_{jj}^T s_{ij} - (1-\gamma_{jj}^T)\hat{\theta}_j) - \lambda_3(p_{-j} - \gamma_{12}^T s_{i(-j)} - (1-\gamma_{12}^T)\hat{\theta}_{-j})}{K_\lambda},\ j \in \{1,2\},$$

$$\gamma_{jj}^T = \frac{K_\lambda F_{-j}\tau_\epsilon}{\lambda_{-j}K_F}, \qquad \gamma_{12}^T = \gamma_{21}^T = \frac{K_\lambda(\tau+\tau_\epsilon)\tau_\epsilon}{K_F}, \qquad a^T = \frac{\lambda_3(\tau+\tau_\epsilon)\tau_\epsilon}{K_F}.$$

Here the weights on the private and public information about the other market, $a^T = \frac{\lambda_3 \gamma_{12}^T}{K_\lambda}$ and $\frac{\lambda_3(1-\gamma_{12}^T)}{K_\lambda}$, sum to the first-best side weight $\frac{\lambda_3}{K_\lambda}$ instead of zero, since even with perfect information the output of one good responds to the cost of the other.

The next proposition separates what the two linkages have in common from what only the cost linkage does.

Proposition 4. *Let the public information be exogenous and $\rho = 0$. Then for any β_3, λ_3:*

(i) the coefficients of the team schedules on the private signals and the team welfare loss $\mathrm{WL}^T = \frac{F_1+F_2}{2(F_1F_2-F_3^2)}$ depend on β_3, λ_3 only through F_3; WL^T is increasing in $|F_3|$, and the side weight a^T is increasing in F_3 and has its sign;

(ii) the main weights factor as

$$\gamma_{jj}^T = \frac{K_\lambda}{\lambda_1\lambda_2}\, g_j(F_3), \qquad g_j(F_3) = \frac{\lambda_j F_{-j}\tau_\epsilon}{F_1F_2 - F_3^2},\ j \in \{1,2\},$$

where g_j is increasing in $|F_3|$, while the factor $\frac{K_\lambda}{\lambda_1\lambda_2} = 1 - \frac{\lambda_3^2}{\lambda_1\lambda_2}$ does not depend on β_3 and is decreasing in $|\lambda_3|$.

Part (i) says that the team does not distinguish between the two structural linkages: its response to private information and its welfare loss are the same under a cost linkage λ_3 and under a demand linkage $\beta_3 = \lambda_3\frac{\tau+\tau_\epsilon}{\tau}$. The cost linkage counts more because it couples the two goods in two places. The demand linkage couples them only through aggregate output, i.e. through prices, so it enters the matrix $\tau\Phi + \tau_\epsilon\Lambda$ only through Φ, which comes from the aggregate part of the losses. The cost linkage also couples the costs of each firm, so it enters through Λ as well, which comes from the distributive part. At this level the two linkages act alike: both of them raise $|F_3|$, and hence the team loss, and both move the side weight in their own direction, which is why the side weights in Theorem 1 behave alike.

When both linkages are present, the two parts also show where they offset each other. At $\beta_3 = -\lambda_3\frac{\tau+\tau_\epsilon}{\tau}$, which requires goods that are substitutes in consumption and complements

in production or the other way round, $F_3 = 0$: the side weight vanishes, so the team uses no private information about the other market, and γ_{jj}^T, which is quasi-convex in β_3, is at its minimum (whenever this value is admissible).

The absolute welfare losses have a common source as well. For the team this is literally the shared channel: by part (i), WL^T sees the linkages only through F_3, it grows in $|F_3|$, and a demand and a cost linkage of opposite signs offset each other in it exactly, so that at $F_3 = 0$ the team loss is that of two independent markets. The reason why the linkages raise the losses is that each of them creates a direction of output change that is resisted only weakly. With substitutes, shifting output from one good to the other barely moves prices (under a demand linkage) or marginal costs (under a cost linkage), because the effect of the rise in one output is offset by the fall in the other; with complements, the same is true of expanding both outputs together. Formally, a linkage of either sign lowers K_ϕ, and the cost linkage also lowers K_λ, so outputs respond strongly to the estimates of the relative cost of the two goods, with substitutes, or of their common cost level, with complements, and an error in these estimates turns into a large output error. This is most transparent in the market equilibrium, where (see Appendix A.3)

$$\mathrm{WL}^M = \frac{\mathrm{tr}(\Lambda^{-1})\tau_\epsilon + \mathrm{tr}(\Phi^{-1})\tau}{2(\tau+\tau_\epsilon)^2}, \qquad \mathrm{tr}(\Lambda^{-1}) = \frac{\lambda_1+\lambda_2}{K_\lambda}, \quad \mathrm{tr}(\Phi^{-1}) = \frac{\phi_1+\phi_2}{K_\phi},$$

so the distributive part grows in $|\lambda_3|$ and the aggregate part in $|\phi_3| = |\beta_3 + \lambda_3|$. In the aggregate part the relevant index is thus ϕ_3 rather than F_3, but it has the same feature: a demand and a cost linkage of opposite signs offset each other.

Part (ii) of Proposition 4, which separates the two linkages, is the subject of the next subsection.

5.2 Where the linkages differ

The two linkages have opposite effects on the main weights, on the welfare losses relative to the first best and on the value of private information for the market. All three differences have the same source: the cost linkage, unlike the demand linkage, also couples the marginal costs inside each firm.

Consider first the main weights. Part (i) of Theorem 1 says that firms rely more on private information as goods become stronger complements or substitutes in consumption. In the formula of Section 5.1 β_3 enters γ_j^T only through $K_F = F_1F_2 - \beta_3^2\tau^2$, which falls in $|\beta_3|$. Economically, the team balances the two sources of losses, and as $|\beta_3|$ increases, the part of the losses that comes from imperfect public information grows, both directly, since $\beta_3\mathbb{E}[(\tilde{x}_1 - x_1^0)(\tilde{x}_2 - x_2^0)]$ grows, and indirectly, since $\phi_j\mathbb{E}[(\tilde{x}_j - x_j^0)^2]$ grows: the term in $\tilde{x}_j - x_j^0$ that depends on the public error of the other market increases in $|\beta_3|$. The team therefore shifts weight toward private information. By Proposition 2, however, it never shifts it beyond the Bayesian weight of the market, so no degree of interaction between goods in consumption can overturn the pecuniary externality of V17 that makes the market overreact.

Now let us switch off the demand linkage and switch on the cost one instead. By part (ii) of Theorem 1 the main weight now falls, and the reason is the coupling channel of Proposition 4. Both linkages carry the public error about the cost of one good into the output of the other, and both let firms hedge against it in the same way, which raises the weight on private information. But a cost linkage also couples the two marginal costs, which has nothing to do with information and lowers this weight. The formula for the cost linkage alone in Section 5.1 makes this explicit: with $\beta_3 = 0$,

$$\gamma_{jj}^T = \left(1 - \frac{\lambda_3^2}{\lambda_1\lambda_2}\right)\frac{\lambda_j F_{-j}\tau_\epsilon}{F_1F_2 - \lambda_3^2(\tau + \tau_\epsilon)^2},$$

where the second factor grows with $|\lambda_3|$ exactly as γ_j^T grows with $|\beta_3|$ under the demand linkage, while the first one, the coupling factor of Proposition 4, has no counterpart there

and falls; by Theorem 1(ii) the coupling factor wins.

Part (ii) of Proposition 4 shows how the coupling channel works: it changes the benchmark against which the main weight is measured. The main weight is a share: the response of the output to the own private signal, $(C^T)_{jj} = \frac{F_{-j}\tau_\epsilon}{K_F}$, divided by the full-information response to the own cost, $(\Lambda^{-1})_{jj} = \frac{\lambda_{-j}}{K_\lambda}$. The numerator sees the linkages only through F_3, and both of them raise it, which gives the factor g_j, the main weight of the demand linkage alone with $\beta_3\tau$ replaced by F_3. The denominator is changed only by the cost linkage, which raises it, since the coupling of marginal costs makes the output respond more strongly to its own cost even under perfect information; this gives the factor $\frac{K_\lambda}{\lambda_1\lambda_2}$, which has nothing to do with information. The market responds to its private signal in proportion to the same benchmark, $(C^M)_{jj} = \gamma^M(\Lambda^{-1})_{jj}$. Hence under the demand linkage the response of the market does not change while that of the team grows, and the gap between them narrows; under the cost linkage both grow, but that of the market grows faster, and the gap widens, since by Theorem 1(ii) the coupling factor always dominates when the cost linkage acts alone.

The second difference concerns the welfare losses. In absolute terms both linkages raise them (Section 5.1), but relative to the value of cost information VI they move them in opposite directions: the demand linkage makes both the market and the team solution relatively better, and the cost linkage makes both of them relatively worse. For the team solution one of the reasons behind this is cross-market hedging, discussed in Section 4.3: while growing interdependence between goods decreases the joint price sensitivity to outputs and hence raises the value of information, the losses from imperfect information grow more slowly in magnitude because of this hedging. But the market solution, which uses the information about the other market only mechanically, shows the same pattern, and the more basic reason is visible in the formula for WL^M of Section 5.1. The value of information VI is proportional to $\mathrm{tr}(\Phi^{-1})$ as well, so the aggregate part of the losses grows at the same rate as VI, and $\frac{\mathrm{WL}^M}{\mathrm{VI}}$ moves only with $K_\phi \mathrm{tr}(\Lambda^{-1})$. The demand linkage lowers K_ϕ without touching Λ, so the ratio falls, while the cost linkage lowers K_λ proportionally more than K_ϕ, so it

rises.

The third difference concerns the value of private information for the market, and it answers a question raised by V17: when is more precise private information harmful for the market? V17 has shown that more precise private information can increase the welfare loss of the market equilibrium, because it increases the response of firms to their private signals, which is already too strong. The team loss never behaves in this way: by Appendix B.3, WL^T is decreasing in τ_ϵ for any values of the parameters, so the possibility of harmful private information is a pure consequence of the pecuniary externality. The linkages decide how large the region where it happens is, and they do it through a single number,

$$R = \frac{\mathrm{tr}(\Lambda^{-1})}{\mathrm{tr}(\Phi^{-1})} = \frac{(\lambda_1 + \lambda_2)K_\phi}{(\phi_1 + \phi_2)K_\lambda},$$

which compares the distributive stakes of the use of information, governed by Λ^{-1}, with its aggregate stakes, governed by Φ^{-1}.

Proposition 5. *Let the public information be exogenous and $\rho = 0$. Then WL^M is increasing in τ_ϵ for $\frac{\tau_\epsilon}{\tau} < 1 - \frac{2}{R}$ and decreasing for $\frac{\tau_\epsilon}{\tau} > 1 - \frac{2}{R}$, so more precise private information can increase WL^M if and only if $R > 2$. Moreover,*

(i) if $\lambda_3 = 0$, then R is an even function decreasing in $|\beta_3|$;

(ii) if $\beta_3 = 0$, then R is an even function increasing in $|\lambda_3|$;

(iii) if $\beta_j = k\lambda_j$ for all $j \in \{1, 2, 3\}$, then $R = 1 + k$ does not depend on the linkages.

So the demand linkage shrinks the region in which more precise private information is harmful for the market, the cost linkage enlarges it, and when the demand matrix is proportional to the cost matrix (Section 5.3) the region does not move at all. For $\beta_3 = \lambda_3 = 0$ the condition $R > 2$ reads $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} > \frac{2}{\phi_1} + \frac{2}{\phi_2}$, which for two identical markets is the condition $\beta > \lambda$ of V17. The same number also governs the relative loss of the market.

Since $\text{VI} = \frac{\text{tr}(\Phi^{-1})}{2\tau_\theta}$,

$$\frac{\text{WL}^M}{\text{VI}} = \frac{\tau_\theta(R\tau_\epsilon + \tau)}{(\tau + \tau_\epsilon)^2},$$

which is the argument above in another form: for the market, the relative-loss parts of Theorem 1 are parts (i) and (ii) above. The same formula shows how bad the market can be: when the public signals add nothing to the prior, $\tau = \tau_\theta$, and $R > 2$, a little private information makes WL^M exceed VI, so the market does worse than it would with prior information only. The team never does, since it can always ignore its signals. In economic terms, interconnection through demand raises the aggregate stakes of getting the outputs right more than their distributive stakes, and so makes the overreaction of the market to private information less costly, while interconnection through costs does the opposite.

Finally, in levels the side weights behave in the same way under both linkages (Proposition 4), but measured against the market they differ as well: under the demand linkage the market ignores the private information about the other market while the team uses it, and under the cost linkage the team uses less of it than the market does (Proposition 3(ii)).

5.3 Aligned linkages

Now let us put the two structural linkages against each other. Recall from Section 2 that the primitives can be collected in the two matrices Λ and B, and notice that Theorem 1 says that the main weight moves in opposite directions in response to the off-diagonal element of each of them. It is therefore natural to ask whether there is a configuration in which these two effects exactly cancel, i.e. in which the coupling channel of Proposition 4 is exactly offset by the hedging channel, and it turns out that there is, and that it has a very simple description.

Theorem 2 (Irrelevance of aligned interconnection)**.** *Let $\rho = 0$ and suppose that the demand matrix is proportional to the cost matrix, i.e. $\exists k > 0$ such that $\beta_j = k\lambda_j$ for all $j \in \{1, 2, 3\}$. Then*

(i) all four team weights coincide,

$$\gamma_{11}^T = \gamma_{22}^T = \gamma_{12}^T = \gamma_{21}^T = \gamma^T = \frac{\tau_\epsilon}{(1+k)\tau + \tau_\epsilon},$$

and the team schedules take the form of the market ones, $X_i^T = \Lambda^{-1}(p - \gamma^T s_i - (1-\gamma^T)\hat{\theta})$ *in vector notation, with* γ^M *replaced by* γ^T*;*

(ii) γ^T*, and hence the wedge* $\gamma^M - \gamma^T$*, does not depend on* λ_3 *(and hence on* $\beta_3 = k\lambda_3$*) at all;*

(iii) $\frac{\mathrm{WL}^T}{\mathrm{VI}}$ *and* $\frac{\mathrm{WL}^M}{\mathrm{VI}}$ *do not depend on* λ_3 *either, while* WL^T *and* WL^M *are increasing in* $|\lambda_3|$ *and unbounded as* $|\lambda_3| \to \sqrt{\lambda_1 \lambda_2}$*.*

I read Theorem 2 as an irrelevance benchmark. Interconnection between the two markets, however strong, has no effect whatsoever on the team-efficient use of decentralized information provided that the demand side and the cost side of that interconnection are aligned in the sense above. Since the market weights never depend on the linkages, along this manifold the whole wedge between the market and the team solution is invariant as well: the information externality is exactly as large in two strongly interconnected markets as it is in two independent ones. Part (i) shows where this comes from: along the manifold the team solution has exactly the shape of the market equilibrium, and the only thing that distinguishes the two is the scalar weight, which is determined by the information parameters and k alone. In the matrix form of Section 4.2 this is the statement that $\tau\Phi + \tau_\epsilon\Lambda = ((1+k)\tau + \tau_\epsilon)\Lambda$, so that C^T is proportional to Λ^{-1}, just as C^M is.

Two restrictions of the theorem are worth emphasizing, since both of them are binding. The first is that $\rho = 0$ is necessary: with correlated public signals the invariance breaks, and for symmetric markets the formulas of Section 6.1 show that γ^T then becomes a linear function of λ_3 (with $\beta_3 = k\lambda_3$), decreasing for $\rho > 0$ and increasing for $\rho < 0$. The second is that the proportionality must hold for the whole matrices. Without it the main weights can still stay at their values for independent markets along a ray of (β_3, λ_3) (Appendix A.4.7),

but this ray depends on the precisions of information, and along it the team solution does not take the form of the market one.

The immediate consequence is that the comparative statics of Theorem 1 can be read as the effects of *departures* from proportionality, and not as the effects of interconnection as such. In particular, the opposite effects of β_3 and λ_3 on the main weights say that departures in the two directions from this manifold are not symmetric. What matters for the use of decentralized information is the misalignment between the demand-side and the cost-side interconnection, and not the magnitude of either of them. The same is true of the value of private information for the market: along the manifold $R = 1 + k$ by Proposition 5(iii), so the region in which more precise private information is harmful does not depend on the degree of interconnection.

The second consequence concerns the measurement of welfare losses, and it settles the question left open after Theorem 1. By part (iii), along the manifold the absolute losses grow without bound as the interconnection becomes stronger, while by parts (i) and (ii) the weights do not move at all. Since the weights are exactly what describes how well the decentralized information is used, we have here a case in which information is demonstrably used equally well at every point, and yet the absolute loss diverges. All of that movement is therefore a pure rescaling of the value of information, which is precisely what $\frac{\text{WL}}{\text{VI}}$ removes and WL does not.

6 The information linkage

Now let us turn to the third linkage. Unlike the demand and the cost linkage, it does not change the payoffs at all: it only makes the errors of the public estimates of the two cost parameters move together. With correlated public signals the team solution does not have a transparent form in general, but for symmetric markets it does, and for any values of all three linkages at once, which is what makes it possible to treat the demand and the cost

linkage on an equal footing here.

Throughout this section I impose homogeneous production and consumption characteristics, i.e. $\lambda_1 = \lambda_2 = \lambda$ *and* $\beta_1 = \beta_2 = \beta$*, while* β_3, λ_3 *and* ρ *are arbitrary. As before,* $\phi = \beta + \lambda$.

6.1 Two V17 economies

With symmetric markets, every cost shock, and every error in its estimate, can be split into a common component, which moves both costs in the same direction, and a relative component, which moves them in opposite directions. The first calls for both outputs to move together, and the second for a reallocation between them. The two components never mix: a common change in the outputs moves both prices and both marginal costs in the same way, a relative change moves them in opposite ways, and the errors of the signals in the two components are uncorrelated for any ρ. Formally, Λ, B and the covariance matrix of the public errors have the same eigenvectors $(1,1)'$ and $(1,-1)'$. Hence the problem of the team splits into two separate problems, one for each component, and each of them is a single-market economy of V17; all three linkages act only by making these two economies different. Let me denote

$$\lambda_\pm = \lambda \pm \lambda_3, \qquad \phi_\pm = \phi \pm (\beta_3 + \lambda_3), \qquad \tau_\pm = \frac{\tau}{1 \pm \rho}, \qquad H_\pm = \tau_\pm \phi_\pm + \tau_\epsilon \lambda_\pm.$$

The two economies have the cost slopes $\lambda_\pm$, the total slopes $\phi_\pm$ and the precisions of public information $\tau_\pm$, and putting their solutions back together gives, for any β_3, λ_3, ρ (the derivation is in Appendix A.4.8),

$$\mathrm{WL}^T = \frac{1}{2}\left(\frac{1}{H_+} + \frac{1}{H_-}\right), \qquad \gamma_{11}^T = \gamma_{22}^T = \frac{K_\lambda \tau_\epsilon}{\lambda}\mathrm{WL}^T,$$

$$a^T = \frac{\tau_\epsilon}{2}\left(\frac{1}{H_-} - \frac{1}{H_+}\right) = \frac{\tau_\epsilon G}{(1-\rho^2)H_+ H_-},$$

where $G = \beta_3\tau + \lambda_3(\tau + (1-\rho^2)\tau_\epsilon) - \rho\phi\tau$.

Each of the three linkages acts on the two V17 economies with opposite signs. Positive structural linkages make the common component steeper and the relative component flatter, $\lambda_+ > \lambda_-$ and $\phi_+ > \phi_-$, while positive correlation makes the public information about the common component worse and about the relative component better, $\tau_+ < \tau_-$. The side weight is the difference between the responses of the team in the two components, and it vanishes exactly when the two components look alike to the team, $H_+ = H_-$, i.e. when $G = 0$; for $\rho = 0$ we have $G = F_3$, the index of Section 5.1.

6.2 Main weights and welfare

The effect of the information linkage itself on the main weights and on the welfare losses need not be monotone. The intuition is that correlation does not change the precision of the public information about each cost, but it redistributes this precision between the two components of Section 6.1: it improves the public information about one of them and worsens it about the other, and it makes the two public errors move together, which is precisely the error that cross-market hedging works against. The losses turn out to be concave in it.

Proposition 6. *Let $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$. Then for any β_3, λ_3, ρ:*

(i) γ^T's and WL^T are concave in ρ;

(ii) γ^T's and WL^T are convex in β_3;

(iii) WL^T is convex in λ_3, while γ^T's are concave in λ_3.

However, the corresponding peak and dip of the functions could be located at the boundary of the domain, i.e. γ^T's and WL^T could be monotone within it.

Parts (ii) and (iii) are the counterparts of Theorem 1 for correlated public signals and for both structural linkages present at once. For $\rho = 0$ and the other structural linkage switched off, the functions are even, so convexity in β_3 means that the main weight increases

in $|\beta_3|$, and concavity in λ_3 means that it decreases in $|\lambda_3|$: the opposite effects of the two linkages in Theorem 1 are the difference between the curvatures of the main weight in them. The welfare loss, in contrast, is convex in both, which is the counterpart of the absolute losses increasing in both linkages. This is the only way in which the two can differ: since $\gamma_{jj}^T = \frac{K_\lambda}{\lambda^2}\lambda\tau_\epsilon \mathrm{WL}^T$, they differ only by the coupling factor of Proposition 4, which depends on neither ρ nor β_3 and falls in $|\lambda_3|$. What correlation changes is the location of the dip and of the peak: for $\rho \neq 0$ they are in general away from zero, and they can be at the boundary.

6.3 Side weights: information against the structural linkages

Recall the side weight a^T from Section 4.2. When $\beta_3 = \lambda_3 = \rho = 0$ it vanishes, and the team solution consists of two copies of the one of V17. By Proposition 4, for $\rho = 0$ it has the sign of F_3, i.e. the sign of the structural linkages, while the information linkage alone gives it the sign of $-\rho$. Hence the information linkage pushes the side weight against each of the structural ones, and the natural question is at which point it exactly offsets each of them. For symmetric markets the answer is a single equation, $G = 0$.

Proposition 7. *Let $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$. Then for any β_3, λ_3, ρ:*

(i) a^T's are increasing in β_3 and in λ_3 and decreasing in ρ; they have the sign of G and vanish exactly when $G = 0$; in particular, $|a^T|$'s are quasi-convex in each of β_3, λ_3 and ρ;

(ii) if $\lambda_3 = 0$, then a^T's vanish at $\beta_3^* = \rho\phi$, which is admissible whenever $|\rho| \leq \frac{\beta}{\phi}$; if $\beta_3 = 0$, then they vanish at $\lambda_3^* = \frac{\rho\phi\tau}{\tau+(1-\rho^2)\tau_\epsilon}$, whenever $|\lambda_3^*| < \lambda$; and if $\rho = 0$, then they vanish at $\beta_3\tau + \lambda_3(\tau + \tau_\epsilon) = 0$.

At $\beta_3 = \beta_3^*$ the team solution uses no information from the other market whatsoever, even though the two markets are strongly connected both in demand and in information. Notice that $\beta_3^* = \rho\phi$ does not depend on τ or τ_ϵ at all: the degree of demand interconnection that neutralizes a given amount of signal correlation is a pure function of market structure.

The reason why β_3^* depends on market structure alone, while λ_3^* also depends on the precisions of information, is the same as in Section 5.1. The demand linkage and the correlation of public signals both act only through the public errors, so the precisions of information cancel out when these two are the only linkages. The cost linkage also couples the costs of each firm, and hence acts through the private signals as well: the more firms rely on them, i.e. the higher $\frac{\tau_\epsilon}{\tau}$, the smaller the cost linkage $|\lambda_3^*|$ that offsets a given correlation.

6.4 Precision of correlated information

Finally, correlation does not change the effect of the precisions of information: as for $\rho = 0$, the main weights and the team loss behave as in V17, and the side weights as in Proposition 3.

Proposition 8. *Let $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$. Then for any β_3, λ_3, ρ:*

(i) γ^T's are increasing in τ_ϵ and decreasing in τ, depending only on $\frac{\tau_\epsilon}{\tau}$;

(ii) WL^T is decreasing in τ, τ_ϵ;

(iii) a^T's depend only on $\frac{\tau_\epsilon}{\tau}$ and have at most one critical point in it; they tend to 0 as $\frac{\tau_\epsilon}{\tau} \to 0$ and to the first-best side weight $\frac{\lambda_3}{K_\lambda}$ as $\frac{\tau_\epsilon}{\tau} \to \infty$. In particular, if $\lambda_3 = 0$ and $\beta_3 \neq \rho\phi$, then $|a^T|$'s are quasi-concave in τ, τ_ϵ and maximized at $\frac{\tau}{\tau_\epsilon} = \lambda\sqrt{\frac{1-\rho^2}{\phi^2-\beta_3^2}}$.

Part (iii) shows the same contrast between the two structural linkages as Proposition 3: the side weight moves from zero, when private information is useless, to the mechanical first-best response to the other cost, when it is perfect. Without a cost linkage both ends are zero and hedging is strongest at an interior precision, which correlation only shifts; with a cost linkage the path ends at the first-best response instead.

6.5 The three linkages compared

Table 1 collects what the three linkages do, each of them switched on alone.

Table 1: The three linkages compared

	Demand ($\lvert\beta_3\rvert \uparrow$)	Cost ($\lvert\lambda_3\rvert \uparrow$)	Information ($\rho \uparrow$)
Main weight γ_{jj}^T	increasing	decreasing	concave
Side weight a^T	increasing, odd	increasing, odd	decreasing
Absolute loss WL^T	increasing	increasing	concave
Relative loss $\mathrm{WL}^T/\mathrm{VI}$	**decreasing**	**increasing**	concave
Reference	Thm. 1(i), Prop. 4	Thm. 1(ii), Prop. 4	Prop. 6, 7

Two things are worth reading off this table. The first is that, in levels, the demand and the cost linkage have opposite effects only on the main weights and the relative losses: in the rows of the side weight and of the absolute loss their columns coincide, which is Proposition 4, although relative to the market they differ on the side weights as well (Proposition 3(ii)). The second is that the information linkage moves the side weight against both structural linkages, so that it can offset either of them, while the two structural linkages can offset each other on the side weight only when they have opposite signs.

This also shows how the linkages offset one another. On the side weights every pair of them can cancel, and for symmetric markets all these points lie on the single plane $G = 0$ of Proposition 7: the information linkage against the demand one at $\beta_3^* = \rho\phi$, against the cost one at λ_3^*, and a demand and a cost linkage of opposite signs at $F_3 = 0$. Only the first of these cancellation points is determined by market structure alone, while the two that involve the cost linkage depend on $\frac{\tau_\epsilon}{\tau}$, because the cost linkage enters the side weight together with the precision of private information. On the main weights, in contrast, the two structural linkages cancel not at a point but along the whole proportionality manifold of Theorem 2, where the team solution collapses to the form of the market one. Taken together, this suggests that the right way to think about interconnection in this model is not as a scalar degree of connection between two markets, but as a vector of channels whose net effect on the use of decentralized information is governed by their mutual alignment.

7 Endogenous public information

Here I consider the case where firms are sophisticated and do take into account the information content of prices, making them the public signals. This case is harder than the exogenous one for a simple reason. In the exogenous framework the three linkages of Section 2.3 are three independent parameters, and this is exactly what made it possible to isolate the effect of each of them in Sections 5 and 6. Once firms learn from prices, this is no longer so: each of the structural linkages generates an endogenous correlation between the two public signals, so that β_3, λ_3 and ρ can no longer be varied separately. In particular, the question of Section 6.3, at which degree of interconnection the correlation of the public signals is exactly offset, no longer has an explicit answer such as $\beta_3^* = \rho\phi$: the correlation that a linkage would have to offset is itself generated by this linkage through the equilibrium strategies, and has to be solved for together with them. As a result, fewer statements can be proven here, and more of them rest on simulations.

The result is a dichotomy, which I state in advance since it organizes the whole section. What survives in full, and can be proven, is the behavior of welfare with respect to the precisions of information: the team-efficient welfare loss is decreasing in every one of them, as in the exogenous case. What does not survive is the behavior of the main weights with respect to the linkages, which carries the opposite effects of Theorem 1: both in the market equilibrium and in the team solution these can be non-monotone, and not even quasi-concave or quasi-convex. Between these two extremes there are a number of regularities that hold in all simulations I have run but that I have not been able to prove.

7.1 Market equilibrium

Since the payoffs and the information structure are symmetric and payoffs are strictly concave, we can WLOG restrict attention to symmetric equilibria. The solution to the problem

of firm i is both unique and symmetric across firms:

$$X_{ij}^M(I_i, p) = \lambda_j^{-1}(p_j - \mathbb{E}[\theta_j|I_i]), j \in \{1, 2\}.$$

Estimation of θ_j with public signals p is tricky. In a linear equilibrium, it can be shown from market clearing that prices p_1, p_2 are linearly related to random variables z_1, z_2 of the form $z_1 = K\theta_1 + \delta_1 u_1 - \delta_3 u_2$, $z_2 = K\theta_2 + \delta_2 u_2 - \delta_4 u_1$, where K, δ_j depend on market parameters β_j and endogenous choices of firms a_{ij}, if they submit schedules of the following form WLOG:

$$X_{i1} = \hat{b}_1 - a_{11}s_{i1} + a_{12}s_{i2} + \hat{c}_1 p_1 + \hat{d}_1 p_2,$$
$$X_{i2} = \hat{b}_2 - a_{22}s_{i2} + a_{21}s_{i1} + \hat{c}_2 p_2 + \hat{d}_2 p_1.$$

One important difference from the model with naive agents and exogenous public information is the endogenous correlation arising between public signals, since both z_1, z_2 linearly depend on u_1, u_2. The reason for this correlation is that each price depends on the market shocks of both goods, through the demand linkage or through the side coefficients that firms choose under either structural linkage. This is exactly the point made above: here the information linkage is not a primitive but a consequence of the structural linkages.

Then, in fact, I could restrict attention to the following alternative form of market schedules, since p_1, p_2 are informationally equivalent to z_1, z_2.

$$X_{i1} = b_1 - a_{11}s_{i1} + a_{12}s_{i2} + c_1 z_1 + d_1 z_2,$$
$$X_{i2} = b_2 - a_{22}s_{i2} + a_{21}s_{i1} + c_2 z_2 + d_2 z_1.$$

So, to find the market equilibrium a_{ij}^M, we need to solve for the linear equilibrium for firms exactly as in the exogenous case with correlated public signals, as if the precision of public information were given. And then we equate the corresponding coefficients for a_{ij} in the two given representations, so that firm aggregate behavior would be the best response to

itself. This fact is at the root of a learning externality: firms fail to account for the impact of their own actions on public information (the price) and hence on other firms. And there is a unique real solution of the complex polynomial system of equations that defines the optimal a_{ij}^M.

Let us call a_{11}^M, a_{22}^M main coefficients, as they correspond to the information about the cost of the produced good in the market schedule of this good itself, while a_{12}^M, a_{21}^M are side coefficients, since they correspond to the information about the cost of the produced good in the market schedule of the other good.

7.2 Team solution

At the team-efficient solution with endogenous public information, expected total surplus is maximized subject to firms using linear output strategies contingent on the information set and prices. It can again be shown that the optimal market schedules have the following form:

$$X_{ij}^T(I_i, p) = \lambda_j^{-1}(p_j - \gamma_j^T s_{ij} - (1-\gamma_j^T)\hat{\theta}_j) + a_j^T(s_{i(-j)} - \hat{\theta}_{-j}), j \in \{1,2\}.$$

In slight contrast with the exogenous case, here $\hat{\theta}_j = \mathbb{E}[\theta_j|z_1, z_2]$, and although z_2 is uncorrelated with θ_1, it is useful in estimation, because z_1 is correlated with z_2 through u_1, u_2 and we can decrease the conditional variance $\mathbb{V}[\theta_j|z_1, z_2] < \mathbb{V}[\theta_j|z_j]$.

The team solution, denoted γ_j^T, a_{ij}^T, minimizes WL, optimally trading off sources of inefficiency in decentralized strategies, taking into account that the precisions of public information depend on a_{ij}, chosen by firms, internalizing possible information externalities.

7.3 Exact results

Although neither of the two systems has a closed-form solution, a few of their properties can be established exactly. The first one is a symmetry. As in the exogenous case, both the market equilibrium and the team solution are invariant under the simultaneous sign change

$(\beta_3, \lambda_3) \to (-\beta_3, -\lambda_3)$ together with $(a_{21}, a_{12}) \to (-a_{21}, -a_{12})$ (Appendices A.5 and A.6). Consequently a_{11}^M, a_{22}^M and γ^T's are even, while a_{21}^M, a_{12}^M and a^T's are odd under this joint sign change; in particular, if $\lambda_3 = 0$ they are respectively even and odd in β_3, and if $\beta_3 = 0$ they are respectively even and odd in λ_3. It is worth being careful here, because the natural stronger statement is false. Parity in β_3 alone requires $\lambda_3 = 0$, and parity in λ_3 alone requires $\beta_3 = 0$; when both linkages are switched on, neither of the two separate parities survives and only the joint one does. The reason is visible in the proof: the quantities M_1, M_2 defined in Appendix A.5 are invariant under a sign change of the side coefficients only if β_3 changes sign at the same time.

For the welfare losses of the team solution it is natural to expect that the team should optimally trade off inefficiencies between different sources of losses and hence could always do better with an increase in any information precision. This intuition turns out to be exactly right, and, in contrast with everything that concerns the linkages, it can be proven in full.

Proposition 9. *In the endogenous public information case,* WL^T *is decreasing in each of* $\tau_\epsilon, \tau_\theta$ *and* τ_u.

The proof, given in Appendix A.6, is an envelope argument combined with the observation that, holding the strategies fixed, an increase in any of the three precisions improves the public estimator in the sense of the positive semidefinite order. It is worth pointing out that the statement about τ_u is not one that I would have expected on the basis of the numerical results, since τ_u influences both the precision of the public signals and the correlation between them, and the team weights themselves can be non-monotone in it. The envelope argument nevertheless goes through, because from the point of view of the team the only thing that matters is the whole conditional variance matrix of the public errors, and that matrix falls with τ_u regardless of what happens to its individual entries.

7.4 Numerical results

The remaining results are of two kinds, and I keep them apart. Statements of the first kind assert that a variable *can* behave in a certain way for some values of the parameters; I state them as Facts, and each of them is established by an explicit example, reported in Appendix C.1 together with the parameter values. Statements of the second kind assert that a variable behaves in a certain way for *all* values of the parameters; these hold in all simulations I have run, but I have not been able to prove them, and I state them as Numerical Propositions.

One of the main features of both the market equilibrium and the team solution is the non-monotone behavior w.r.t. τ_u, but there is a rather convincing explanation. It influences both the precision of public signals and the correlation between them, while we have seen that in the exogenous case the effect of correlation was non-monotone. Hence its joint effect together with the precision of public information, which is moreover endogenously determined, should be expected to be ambiguous. For this reason I do not comment on the behavior of the variables w.r.t. τ_u any further, and none of the figures in Appendix C.2 varies it.

When it comes to the effect of the linkages on the main coefficients, practically nothing could be determined, and this is the fragility announced at the start of the section. The main reason behind that is the aforementioned endogenous correlation between public signals that depends on β_3 and λ_3, while for the exogenous case we have seen in Section 6.3 that each of them is a counter-current force to ρ, and hence their joint effect in the endogenous case becomes very ambiguous. For the side coefficients of the team solution the effects of β_3 and λ_3 are unknown for a similar reason, since in the exogenous case the direction of the correlation effect was opposite to the direction of each of them.

Fact 1. *In the endogenous public information case there exist values of the parameters for which, in the market equilibrium,*

(i) a_{11}^M, a_{22}^M *are non-monotone in* β_3*, and neither quasi-concave nor quasi-convex in it; and*

the same is true for λ_3

(ii) $|a_{21}^M|, |a_{12}^M|$ *are non-monotone in* τ_ϵ

(iii) WL^M *is non-monotone in* τ_ϵ

(iv) if $\lambda_3 = 0$, WL^M *is non-monotone in* $|\beta_3|$

Fact 2. *In the endogenous public information case there exist values of the parameters for which, in the team solution,*

(i) γ^T*'s and* a^T*'s are non-monotone in* β_3*, and neither quasi-concave nor quasi-convex in it; and the same is true for* λ_3

(ii) γ^T*'s are non-monotone in* τ_θ *and in* τ_ϵ

(iii) a^T*'s are non-monotone in* τ_θ *and in* τ_ϵ

Part (iii) of Fact 1 is the counterpart of Proposition 5, and the reason for it is the same: an overreaction to private information that becomes more costly as private information becomes more precise. Part (ii) of Fact 2 is much rarer than the others: in my simulations the main weights of the team are decreasing in τ_θ and increasing in τ_ϵ for the great majority of parameter values, and in the examples of Figure 7 the non-monotonicity takes the form of a sharp drop of one of the weights followed by a recovery.

However, when it comes to the behavior with respect to the precision of information, things are much nicer for the market equilibrium: more precise private information $\Rightarrow$ firms rely more on it, more precise public $\Rightarrow$ firms rely less on it. Here we need to remember that the precision of public information positively depends on τ_θ, τ_u, but, as we have seen, τ_u also affects the correlation between public signals, and that is why it is excluded from the following statement.

Numerical Proposition 1. *In the endogenous public information case,* a_{11}^M, a_{22}^M *are decreasing in* τ_θ *and increasing in* τ_ϵ*, and* $|a_{21}^M|, |a_{12}^M|$ *are decreasing in* τ_θ*.*

For the side coefficients of the market equilibrium the results are more well-behaved w.r.t. the linkages, since both these coefficients and the linkages correspond to the degree of connection between the two markets. However, unlike in the exogenous case, the side coefficients in the market equilibrium seem to be monotonic w.r.t. τ_θ, and I do not have a reasonable explanation for that. Notice also that if $\lambda_3 = 0$, then $\frac{a_{12}^M}{a_{21}^M} = \frac{\lambda_2}{\lambda_1}$, since the weights of market firms on their private signals are Bayesian and the private signals about the two costs are equally precise (Appendix A.5), so it is enough to establish the next statement for one of the two side coefficients.

Numerical Proposition 2. *In the endogenous public information case, if $\lambda_3 = 0$, then a_{21}^M, a_{12}^M are increasing in β_3; and if $\beta_3 = 0$, then they are increasing in λ_3.*

As for welfare losses, for the demand linkage the team loss seems to be repeating the pattern of the exogenous case, increasing in $|\beta_3|$, while the market loss need not, by Fact 1(iv). For the cost linkage the pattern is different for both: instead of increasing in $|\lambda_3|$ as in Theorem 1(ii), the losses first increase and then decrease as $|\lambda_3|$ approaches its upper bound.

Numerical Proposition 3. *In the endogenous public information case the following is true:*

(i) if $\lambda_3 = 0$, then WL^T is increasing in $|\beta_3|$

(ii) if $\beta_3 = 0$, then $\mathrm{WL}^M, \mathrm{WL}^T$ are quasi-concave in λ_3 on intervals where either $\lambda_3 \geq 0$ or $\lambda_3 \leq 0$

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A Proofs, derivations and supplemental equations

Structure of Appendix A

Subsection A.1 contains all derivations and supplementary equations for the team solution with exogenous public information, correlated costs and complements/substitutes in production.

Subsection A.2 provides exact equations for the estimators that are used in Section 6: the exogenous information with correlated costs case.

Subsection A.3 contains the matrix representation of the welfare losses, which is used in several proofs.

Subsection A.4 contains all proofs for Sections 4, 5 and 6, i.e. of Propositions 2–8 and of Theorems 1 and 2, together with the derivation of the team solution for symmetric markets.

Subsection A.5 contains all derivations and supplementary equations for the market equilibrium with endogenous public information, together with the proof of the joint parity for it.

Subsection A.6 contains all derivations and supplementary equations for the team solution with endogenous public information, together with the proof of the joint parity for it and the proof of Proposition 9.

Appendix B contains the results that carry over from V17 or from the single-good analysis: Cournot competition, heterogeneous precisions and the comparative statics in the precisions of information.

A.1 General team solution with exogenous information

Here and below I use the notation of Sections 2 and 4.2: $F_j = \phi_j\tau + \lambda_j\tau_\epsilon$, $j \in \{1,2,3\}$, and $K_x = x_1x_2 - x_3^2$ for $x \in \{\lambda, \beta, \phi, F\}$. Notice that when $\lambda_3 = 0$, which is the case for most of this subsection, this gives $K_\phi = \phi_1\phi_2 - \beta_3^2$ and $K_F = F_1F_2 - (\beta_3\tau)^2$.

First of all let us consider WLOG the following market schedules

$$X_{i1} = b_1 - a_{11}s_{i1} + a_{12}s_{i2} + c_1\hat{\theta}_1 + d_1\hat{\theta}_2 + e_1p_1 + f_1p_2$$
$$X_{i2} = b_2 - a_{22}s_{i2} + a_{21}s_{i1} + c_2\hat{\theta}_2 + d_2\hat{\theta}_1 + e_2p_2 + f_2p_1$$

It can be easily checked that in general the maximization of $\mathbb{E}[\mathrm{TS}]$ is a convex quadratic problem in the coefficients of these equations (the best way to see this is to notice that WL is a convex function of a linear combination of parameters, being a sum of two quadratic forms, being positive definite), so I can restrict attention to F.O.C.s, which should give a

unique optimum:

$$\frac{\partial \mathbb{E}[\text{TS}]}{\partial e_1} = \mathbb{E}[(\frac{\partial \text{CS}}{\partial \tilde{x}_1} - \frac{\partial C(x_{i1}, x_{i2})}{\partial x_{i1}})p_1] = \mathbb{E}[(p_1 - \text{MC}_1(x_{i1}, x_{i2}))p_1] = 0$$

Using the same trick we can establish that if we take all the derivatives except w.r.t a_{ij}'s we have the following system

$$\mathbb{E}[p_j - \text{MC}_j(x_{i1}, x_{i2})] = 0, j \in \{1, 2\}$$
$$\mathbb{E}[(p_j - \text{MC}_j(x_{i1}, x_{i2}))p_k] = 0, j, k \in \{1, 2\}$$
$$\mathbb{E}[(p_j - \text{MC}_j(x_{i1}, x_{i2}))\hat{\theta}_k] = 0, j, k \in \{1, 2\}$$

From the first two groups of equations, since we have a linear-Gaussian model, we can obtain that

$$\mathbb{E}[p_j - \text{MC}_j(x_{i1}, x_{i2})|p_1, p_2] = 0, j \in \{1, 2\}$$

Using here the fact that for naive agents $\mathbb{E}[p_j\theta_k] = 0, j, k \in \{1, 2\}$ we can derive

$$\mathbb{E}[X_{i1}|p_1, p_2] = b_1 + e_1p_1 + f_1p_2$$
$$\mathbb{E}[X_{i2}|p_1, p_2] = b_2 + e_2p_2 + f_2p_1$$

Plugging this into MC above means that we should have $\lambda_j b_j = (1 - \lambda_j e_j) = \lambda_j f_j = 0$

For the third group of equations, using the fact that $\mathbb{E}[p_j z_k] = 0, j, k \in \{1, 2\}$ and hence $\mathbb{E}[p_j\hat{\theta}_k] = 0, j, k \in \{1, 2\}$ since firms are naive we can establish that

$$\mathbb{E}[(\theta_1 + \lambda_1(-a_{11}\theta_1 + a_{12}\theta_2 + c_1\hat{\theta}_1 + d_1\hat{\theta}_2))\hat{\theta}_j] = 0, j \in \{1, 2\}$$
$$\mathbb{E}[(\theta_2 + \lambda_2(-a_{22}\theta_2 + a_{21}\theta_1 + c_2\hat{\theta}_2 + d_2\hat{\theta}_1))\hat{\theta}_j] = 0, j \in \{1, 2\}$$

Using the BLP property: $\mathbb{E}[(\theta_j - \hat{\theta}_j)\hat{\theta}_k] = 0, j, k \in \{1, 2\}$ from here we can establish that

$$1 - \lambda_1 c_1 - \lambda_1 a_{11} = 0;\ \lambda_1 d_1 + \lambda_1 a_{12} = 0$$

$$1 - \lambda_2 c_2 - \lambda_2 a_{22} = 0;\ \lambda_2 d_2 + \lambda_2 a_{21} = 0$$

Hence if we plug all of these back into the schedules, we will indeed get the following form:

$$X_{ij}^T(I_i, p) = \lambda_j^{-1}(p_j - \gamma_j^T s_{ij} - (1 - \gamma_j^T)\hat{\theta}_j) + a_j^T(s_{i(-j)} - \hat{\theta}_{-j}), j \in \{1, 2\}$$

Then we can use this form to establish that $x_{ij} - \tilde{x}_j = a_j^T \epsilon_{i(-j)} - \lambda_j^{-1}\gamma_j^T \epsilon_{ij}$

While for $\tilde{x}_j - x_j^0$, we first compute $\tilde{x}_j$ in equilibrium, and then it can be easily found that:

$$\tilde{x}_j - x_j^0 = \frac{\beta_3[(1 - \gamma_{-j}^T)(\hat{\theta}_{-j} - \theta_{-j}) + \lambda_{-j} a_{-j}^T(\hat{\theta}_j - \theta_j)] - \phi_{-j}[(1 - \gamma_j^T)(\hat{\theta}_j - \theta_j) + \lambda_j a_j^T(\hat{\theta}_{-j} - \theta_{-j})]}{K_\phi}$$

Let us notice that $\mathbb{E}[\epsilon_{ij}^2] = \tau_\epsilon^{-1}, \mathbb{E}[(\hat{\theta}_j - \theta_j)^2] = \tau^{-1}, j \in \{1, 2\}, \mathbb{E}[(\hat{\theta}_1 - \theta_1)(\hat{\theta}_2 - \theta_2)] = \rho\tau^{-1}$

Then I plug these formulas into the equation for welfare losses from Section 3 to obtain the following quadratic losses:

$$\mathrm{WL}^T(\gamma_1, \gamma_2, a_2, a_1) = \frac{\gamma_1^2}{2\lambda_1\tau_\epsilon} + \frac{\gamma_2^2}{2\lambda_2\tau_\epsilon} + \frac{\lambda_2 a_2^2}{2\tau_\epsilon} + \frac{\lambda_1 a_1^2}{2\tau_\epsilon} +$$

$$+\frac{\phi_2}{2K_\phi\tau}(1 - \gamma_1)^2 - \frac{\beta_3\lambda_2}{K_\phi\tau}(1 - \gamma_1)a_2 + \frac{\lambda_2^2\phi_1}{2K_\phi\tau}a_2^2 + \frac{\phi_1}{2K_\phi\tau}(1 - \gamma_2)^2 - \frac{\beta_3\lambda_1}{K_\phi\tau}(1 - \gamma_2)a_1 + \frac{\lambda_1^2\phi_2}{2K_\phi\tau}a_1^2 -$$

$$-\rho[\frac{\beta_3}{K_\phi\tau}(1 - \gamma_1)(1 - \gamma_2) + \frac{\beta_3\lambda_1\lambda_2}{K_\phi\tau}a_2 a_1 + \frac{\lambda_1\phi_2}{K_\phi\tau}(1 - \gamma_1)a_1 + \frac{\lambda_2\phi_1}{K_\phi\tau}(1 - \gamma_2)a_2]$$

Here I intentionally omit the algebra of F.O.C.s, but all the following statements can be easily checked:

1)With $\rho = 0$ the answer is: $\gamma_1^T = \lambda_1 \frac{F_2}{K_F}\tau_\epsilon, \gamma_2^T = \lambda_2 \frac{F_1}{K_F}\tau_\epsilon, a_1^T = a_2^T = \frac{\beta_3 \tau}{K_F}\tau_\epsilon$

2)With $\rho \neq 0$ things are going to be much more complex

$K_{\rho+} = (\phi_1\tau + \lambda_1\tau_\epsilon(1+\rho))(\phi_2\tau + \lambda_2\tau_\epsilon(1+\rho)) - (\beta_3\tau)^2$

$K_{\rho-} = (\phi_1\tau + \lambda_1\tau_\epsilon(1-\rho))(\phi_2\tau + \lambda_2\tau_\epsilon(1-\rho)) - (\beta_3\tau)^2$

$F_{23\rho+} = \phi_2\tau + \lambda_2\tau_\epsilon(1+\rho) - \beta_3\tau,\ F_{23\rho-} = \phi_2\tau + \lambda_2\tau_\epsilon(1-\rho) + \beta_3\tau$

$F_{13\rho+} = \phi_1\tau + \lambda_1\tau_\epsilon(1+\rho) - \beta_3\tau,\ F_{13\rho-} = \phi_1\tau + \lambda_1\tau_\epsilon(1-\rho) + \beta_3\tau$

$$\gamma_1^T = 0.5\lambda_1\left[\frac{(1-\rho)F_{23\rho-}}{K_{\rho-}} + \frac{(1+\rho)F_{23\rho+}}{K_{\rho+}}\right]\tau_\epsilon, \gamma_2^T = 0.5\lambda_2\left[\frac{(1-\rho)F_{13\rho-}}{K_{\rho-}} + \frac{(1+\rho)F_{13\rho+}}{K_{\rho+}}\right]\tau_\epsilon$$

$$a_1^T = 0.5\left[\frac{(1-\rho)F_{23\rho-}}{K_{\rho-}} - \frac{(1+\rho)F_{23\rho+}}{K_{\rho+}}\right]\tau_\epsilon, a_2^T = 0.5\left[\frac{(1-\rho)F_{13\rho-}}{K_{\rho-}} - \frac{(1+\rho)F_{13\rho+}}{K_{\rho+}}\right]\tau_\epsilon$$

3)If $\rho = 0$ from the general formula for welfare losses I can establish that:

$$\mathrm{WL}^M = \frac{(\lambda_1+\lambda_2)\tau_\epsilon}{2\lambda_1\lambda_2(\tau+\tau_\epsilon)^2} + \frac{(\phi_1+\phi_2)\tau}{2K_\phi(\tau+\tau_\epsilon)^2}; \mathrm{WL}^T = \frac{F_1+F_2}{2K_F}$$

By the definition in Section 3,

$$\mathrm{VI} = \frac{\mathrm{tr}(\Phi^{-1})}{2\tau_\theta} = \frac{\phi_1+\phi_2}{2\tau_\theta K_\phi}$$

$$\mathrm{WL}^T = \frac{(1-\gamma_1^T)(\phi_2-\rho\beta_3) + (1-\gamma_2^T)(\phi_1-\rho\beta_3) + \lambda_2 a_2^T(\rho\phi_1-\beta_3) + \lambda_1 a_1^T(\rho\phi_2-\beta_3)}{2K_\phi\tau}$$

4)In the symmetric case $\lambda_1 = \lambda_2 = \lambda, \beta_1 = \beta_2 = \beta,\ \lambda+\beta = \phi$ (recall $F = \phi\tau + \lambda\tau_\epsilon$) if $\rho \neq 0$:

$$\gamma^T = \frac{\lambda[F - \beta_3\tau\rho - \lambda\tau_\epsilon\rho^2]\tau_\epsilon}{(F-\beta_3\tau-\lambda\tau_\epsilon\rho)(F+\beta_3\tau+\lambda\tau_\epsilon\rho)}$$

$$a^T = \frac{(\beta_3-\rho\phi)\tau\tau_\epsilon}{(F-\beta_3\tau-\lambda\tau_\epsilon\rho)(F+\beta_3\tau+\lambda\tau_\epsilon\rho)}$$

$$\mathrm{WL}^T = \frac{F-\beta_3\tau\rho-\lambda\tau_\epsilon\rho^2}{(F-\beta_3\tau-\lambda\tau_\epsilon\rho)(F+\beta_3\tau+\lambda\tau_\epsilon\rho)} = \frac{\gamma^T}{\lambda\tau_\epsilon}$$

In the general case when $\lambda_3 \neq 0$ the following is true (recall that here $K_\phi = \phi_1\phi_2 - \phi_3^2$):

$$\mathrm{WL}(a_{11}, a_{22}, a_{21}, a_{12}) = a_{11}^2 \left(\frac{\lambda_1}{2\tau_\epsilon} + \frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{2K_\phi \tau} \right) + a_{22}^2 \left(\frac{\lambda_2}{2\tau_\epsilon} + \frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{2K_\phi \tau} \right)$$

$$+ a_{12}^2 \left(\frac{\lambda_1}{2\tau_\epsilon} + \frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{2K_\phi \tau} \right) + a_{21}^2 \left(\frac{\lambda_2}{2\tau_\epsilon} + \frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{2K_\phi \tau} \right)$$

$$- a_{11}a_{21} \left(\frac{\lambda_3}{\tau_\epsilon} + \frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau} \right) - a_{22}a_{12} \left(\frac{\lambda_3}{\tau_\epsilon} + \frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau} \right)$$

$$- \rho a_{11}a_{12} \frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{K_\phi \tau} - \rho a_{22}a_{21} \frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{K_\phi \tau}$$

$$+ \rho(a_{11}a_{22} + a_{12}a_{21}) \frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau} + \frac{\phi_1 + \phi_2 - 2\rho\phi_3}{2K_\phi \tau}$$

$$- \frac{a_{11}}{K_\phi \tau} \left(\rho(\lambda_3\beta_1 - \beta_3\lambda_1) + \lambda_1\phi_2 - \lambda_3\phi_3 \right) - \frac{a_{22}}{K_\phi \tau} \left(\rho(\lambda_3\beta_2 - \beta_3\lambda_2) + \lambda_2\phi_1 - \lambda_3\phi_3 \right)$$

$$+ \frac{a_{21}}{K_\phi \tau} \left(\rho(\lambda_2\phi_1 - \lambda_3\phi_3) + \lambda_3\beta_2 - \beta_3\lambda_2 \right) + \frac{a_{12}}{K_\phi \tau} \left(\rho(\lambda_1\phi_2 - \lambda_3\phi_3) + \lambda_3\beta_1 - \beta_3\lambda_1 \right)$$

Here a_{jk} are the coefficients of the schedules at the beginning of this subsection, so that comparing them with the schedules of Section 4.2 gives $a_{11} = \frac{\lambda_2\gamma_{11}}{K_\lambda}, a_{22} = \frac{\lambda_1\gamma_{22}}{K_\lambda}, a_{21} = \frac{\lambda_3\gamma_{21}}{K_\lambda}, a_{12} = \frac{\lambda_3\gamma_{12}}{K_\lambda}$, $K_x = x_1x_2 - x_3^2, x \in \{\lambda, \beta, \phi\}$

A.2 Estimators with exogenous information and correlated costs

In the market equilibrium I have the estimator $\tilde{\theta}_1 = \mathbb{E}[\theta_1 | I_i]$ of the following form

1)If $cov(\theta_1, \theta_2) = \rho'\tau_\theta^{-1}, \omega_1 \perp \omega_2$, $\tilde{\theta}_1 = as_{i1} + bs_{i2} + cz_1 + dz_2$;

$$a = \frac{\tau_\epsilon((\tau_\omega + \tau_\epsilon)(1 - \rho'^2) + \tau_\theta)}{\tau_\theta^2 + 2\tau_\theta(\tau_\omega + \tau_\epsilon) + (\tau_\omega + \tau_\epsilon)^2(1 - \rho'^2)}, b = \frac{\rho'\tau_\epsilon\tau_\theta}{\tau_\theta^2 + 2\tau_\theta(\tau_\omega + \tau_\epsilon) + (\tau_\omega + \tau_\epsilon)^2(1 - \rho'^2)}$$

$$c = \frac{\tau_\omega((\tau_\omega + \tau_\epsilon)(1 - \rho'^2) + \tau_\theta)}{\tau_\theta^2 + 2\tau_\theta(\tau_\omega + \tau_\epsilon) + (\tau_\omega + \tau_\epsilon)^2(1 - \rho'^2)}, d = \frac{\rho'\tau_\omega\tau_\theta}{\tau_\theta^2 + 2\tau_\theta(\tau_\omega + \tau_\epsilon) + (\tau_\omega + \tau_\epsilon)^2(1 - \rho'^2)}$$

2)If $cov(\omega_1, \omega_2) = \rho'\tau_\omega^{-1}, \theta_1 \perp \theta_2$, $\tilde{\theta}_1 = as_{i1} + bs_{i2} + cz_1 + dz_2$;

$$a = \frac{\tau_\epsilon(\tau_\omega + (\tau_\theta + \tau_\epsilon)(1-\rho'^2))}{\tau_\omega^2 + (\tau_\theta+\tau_\epsilon)^2(1-\rho'^2) + \tau_\omega(\tau_\theta+\tau_\epsilon)(2-\rho'^2)}, b = \frac{\rho'\tau_\omega\tau_\epsilon}{\tau_\omega^2 + (\tau_\theta+\tau_\epsilon)^2(1-\rho'^2) + \tau_\omega(\tau_\theta+\tau_\epsilon)(2-\rho'^2)}$$

$$c = \frac{\tau_\omega(\tau_\omega + \tau_\theta + \tau_\epsilon)}{\tau_\omega^2 + (\tau_\theta+\tau_\epsilon)^2(1-\rho'^2) + \tau_\omega(\tau_\theta+\tau_\epsilon)(2-\rho'^2)}, d = \frac{-\rho'\tau_\omega(\tau_\theta+\tau_\epsilon)}{\tau_\omega^2 + (\tau_\theta+\tau_\epsilon)^2(1-\rho'^2) + \tau_\omega(\tau_\theta+\tau_\epsilon)(2-\rho'^2)}$$

And for the team solution the estimator $\hat{\theta}_1$ based on public information is then the following:

1)If $cov(\theta_1, \theta_2) = \rho'\tau_\theta^{-1}, \omega_1 \perp \omega_2$, $\hat{\theta}_1 = az_1 + bz_2$;

$$a = \frac{\tau_\omega^2 + \tau_\theta\tau_\omega(1-\rho'^2)}{\tau_\omega^2 + \tau_\theta^2 + \tau_\omega\tau_\theta(2-\rho'^2)}, b = \frac{\rho'\tau_\omega\tau_\theta}{\tau_\omega^2 + \tau_\theta^2 + \tau_\omega\tau_\theta(2-\rho'^2)}$$

$$\text{And } V(\theta_j|\hat{\theta}_j) = V(\theta_j - \hat{\theta}_j) = \frac{\tau_\omega + \tau_\theta(1-\rho'^2)}{\tau_\omega^2 + \tau_\theta^2 + \tau_\omega\tau_\theta(2-\rho'^2)} = \tau^{-1}$$

$$Cov(\theta_1 - \hat{\theta}_1, \theta_2 - \hat{\theta}_2) = \frac{\rho'((1-a)^2 + b^2) - 2(1-a)b}{\tau_\theta} + \frac{2ab}{\tau_\omega} = \rho\tau^{-1}$$

2)If $cov(\omega_1, \omega_2) = \rho'\tau_\omega^{-1}, \theta_1 \perp \theta_2$, $\hat{\theta}_1 = az_1 + bz_2, \hat{\theta}_2 = az_2 + bz_1$;

$$a = \frac{\tau_\omega(\tau_\omega + \tau_\theta)}{(\tau_\omega + \tau_\theta)^2 - \rho'^2\tau_\theta}, b = \frac{-\rho\tau_\omega\tau_\theta}{(\tau_\omega + \tau_\theta)^2 - \rho'^2\tau_\theta}$$

$$\text{And } V(\theta_j|\hat{\theta}_j) = V(\theta_j - \hat{\theta}_j) = \frac{\tau_\omega + \tau_\theta(1-\rho'^2)}{(\tau_\omega + \tau_\theta)^2 - \rho'^2\tau_\theta} = \tau^{-1}$$

$$Cov(\theta_1 - \hat{\theta}_1, \theta_2 - \hat{\theta}_2) = \frac{2ab + \rho'(a^2 + b^2)}{\tau_\omega} - \frac{2(1-a)b}{\tau_\theta} = \rho\tau^{-1}$$

All results for $\tilde{\theta}_2, \hat{\theta}_2$ in this subsection obviously follow from the symmetry of the model.

A.3 Matrix representation of the welfare losses

Let me write the team schedules of Section 4.2 in the matrix form

$$X_i = \Lambda^{-1}p - Cs_i - D\hat{\theta}, \qquad C + D = \Lambda^{-1},$$

where $s_i = (s_{i1}, s_{i2})'$, $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)'$ and $C_{jk} = (\Lambda^{-1})_{jk}\gamma_{jk}$, while for $\lambda_3 = 0$ the off-diagonal entries are $C_{(-j)j} = -D_{(-j)j} = -a_{-j}$. The argument of Subsection A.1 that gives this form, i.e. the F.O.C.s with respect to the coefficients on the prices and on the public signals, uses neither the homogeneity of precisions nor $\rho = 0$, and the market schedules have the same form with $C = C^M$.

Let $\varepsilon_i = s_i - \theta$ and $e = \hat{\theta} - \theta$, with covariance matrices Σ_ε and Σ_e. Then $x_i - \tilde{x} = -C\varepsilon_i$ and $\tilde{x} = \Lambda^{-1}(p - \theta) - De$, while $x^0 = \Lambda^{-1}(p^0 - \theta)$ and $p - p^0 = -B(\tilde{x} - x^0)$, hence $\Lambda^{-1}\Phi(\tilde{x} - x^0) = -De$ and $\tilde{x} - x^0 = -\Phi^{-1}\Lambda De$. So the welfare losses of Section 3 are

$$\mathrm{WL} = \frac{1}{2}\mathbb{E}[(x_i - \tilde{x})'\Lambda(x_i - \tilde{x})] + \frac{1}{2}\mathbb{E}[(\tilde{x} - x^0)'\Phi(\tilde{x} - x^0)] = \frac{1}{2}\mathrm{tr}(C'\Lambda C\Sigma_\varepsilon) + \frac{1}{2}\mathrm{tr}(D'\Lambda\Phi^{-1}\Lambda D\Sigma_e).$$

In the homogeneous case $\Sigma_\varepsilon = \tau_\epsilon^{-1} I$ and $\Sigma_e = \tau^{-1}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, and if $\rho = 0$ but the precisions differ, $\Sigma_\varepsilon = \mathrm{diag}(\tau_{\epsilon 1}^{-1}, \tau_{\epsilon 2}^{-1})$ and $\Sigma_e = \mathrm{diag}(\tau_1^{-1}, \tau_2^{-1})$, so that

$$\mathrm{WL} = \frac{1}{2}\sum_{k=1}^{2}\left(\frac{c_k'\Lambda c_k}{\tau_{\epsilon k}} + \frac{d_k'\Lambda\Phi^{-1}\Lambda d_k}{\tau_k}\right),$$

where c_k and d_k are the k-th columns of C and D, i.e. the coefficients of both schedules on the private and on the public signal about θ_k, and d_k is the k-th column of Λ^{-1} minus c_k.

Since the first term is a positive definite quadratic form in C, the team problem has a unique solution, given by the F.O.C. $\Lambda C\Sigma_\varepsilon = \Lambda\Phi^{-1}\Lambda D\Sigma_e$. Plugging it into the second term gives $\mathrm{tr}(D'\Lambda\Phi^{-1}\Lambda D\Sigma_e) = \mathrm{tr}(D'\Lambda C\Sigma_\varepsilon)$, so that at the team solution

$$\mathrm{WL}^T = \frac{1}{2}\mathrm{tr}((C + D)'\Lambda C\Sigma_\varepsilon) = \frac{1}{2}\mathrm{tr}(C^T\Sigma_\varepsilon),$$

i.e. $\mathrm{WL}^T = \frac{\mathrm{tr}(C^T)}{2\tau_\epsilon}$ with homogeneous private precision. For $\rho = 0$ this gives $\mathrm{WL}^T = \frac{1}{2}\mathrm{tr}((\tau\Phi + \tau_\epsilon\Lambda)^{-1}) = \frac{F_1 + F_2}{2K_F}$, which for $\lambda_3 = 0$ is the formula of Subsection A.1.

For homogeneous precisions and $\rho = 0$ the same representation with $C = \gamma^M\Lambda^{-1}$, $D =$

$(1-\gamma^M)\Lambda^{-1}$ gives the market losses

$$\mathrm{WL}^M = \frac{(\gamma^M)^2\mathrm{tr}(\Lambda^{-1})}{2\tau_\epsilon} + \frac{(1-\gamma^M)^2\mathrm{tr}(\Phi^{-1})}{2\tau} = \frac{\mathrm{tr}(\Lambda^{-1})\tau_\epsilon + \mathrm{tr}(\Phi^{-1})\tau}{2(\tau+\tau_\epsilon)^2}.$$

A.4 Proofs for exogenous public information

A.4.1 Proof of Proposition 2

I prove the statement for $j=1$; for $j=2$ it follows from the symmetry of the model. First of all let us notice that F_1, F_2, F_3 are the entries of the matrix $\tau\Phi + \tau_\epsilon\Lambda$, which is positive definite, hence $F_1, F_2, K_F > 0$ and

$$\gamma_1^M \geq \gamma_{11}^T \Leftrightarrow \lambda_2 K_F \geq K_\lambda F_2(\tau+\tau_\epsilon).$$

Direct computation gives

$$\lambda_2 K_F - K_\lambda F_2(\tau+\tau_\epsilon) = \tau\left[(\tau+\tau_\epsilon)(\lambda_2^2\beta_1 - 2\lambda_2\lambda_3\beta_3 + \lambda_3^2\beta_2) + \lambda_2\tau K_\beta\right].$$

The first term in the brackets is $(\tau+\tau_\epsilon)v'Bv$ with $v=(\lambda_2,-\lambda_3)'$, and it is nonnegative, since B is positive semidefinite because of $\beta_3^2 \leq \beta_1\beta_2$, which also gives $K_\beta \geq 0$ for the second term. Moreover, if $\beta_3^2 < \beta_1\beta_2$, then the second term is strictly positive, while if $\lambda_3 = 0$, then the first one is $(\tau+\tau_\epsilon)\lambda_2^2\beta_1 > 0$, hence proven.

In particular, for $\lambda_3 = 0$ the bracket equals $\lambda_2[(\beta_1\phi_2 - \beta_3^2)\tau + \beta_1\lambda_2\tau_\epsilon]$, which is obviously positive, since $\beta_3^2 \leq \beta_1\beta_2 < \beta_1\phi_2$. In general the inequality can be an equality only if $\beta_3^2 = \beta_1\beta_2$ and $\lambda_2\beta_1 = \lambda_3\beta_3$, i.e. when v is in the kernel of B, and it cannot be an equality for $j=1$ and $j=2$ at once, since together with $\lambda_1\beta_2 = \lambda_3\beta_3$ this would give $\lambda_1\lambda_2 = \lambda_3^2$.

Let me also notice that the weak inequality follows at once from the matrix form of Section 4.2: since $\tau\Phi + \tau_\epsilon\Lambda = (\tau+\tau_\epsilon)\Lambda + \tau B \succeq (\tau+\tau_\epsilon)\Lambda$, the matrix $C^M - C^T$ is positive semidefinite, and $\gamma_j^M - \gamma_{jj}^T = \frac{K_\lambda}{\lambda_{-j}}(C^M - C^T)_{jj} \geq 0$ as a diagonal entry of it.

A.4.2 Proof of Proposition 3

i)

$$(|a_1^T|)^{-1} = (|a_2^T|)^{-1} = \frac{(\phi_1\tau + \lambda_1\tau_\epsilon)(\phi_2\tau + \lambda_2\tau_\epsilon) - (\beta_3\tau)^2}{|\beta_3|\tau\tau_\epsilon} = \frac{K_\phi\tau}{|\beta_3|\tau_\epsilon} + \frac{\lambda_1\lambda_2\tau_\epsilon}{|\beta_3|\tau} + \frac{\phi_1\lambda_2 + \phi_2\lambda_1}{|\beta_3|}$$

This function is definitely quasi-convex in τ, τ_ϵ, hence $|a^T|$ are quasi-concave

From the AM-GM inequality it can be easily seen that this expression is minimized and hence a_1^T, a_2^T are maximized when $(\frac{\tau}{\tau_\epsilon})^2 = \frac{\lambda_1\lambda_2}{K_\phi}$ hence proven

ii)With $\beta_3 = 0$ we have $F_3 = \lambda_3(\tau + \tau_\epsilon)$, and dividing by τ^2 and denoting $r = \frac{\tau_\epsilon}{\tau}$ we get

$$a^T = \frac{F_3\tau_\epsilon}{F_1F_2 - F_3^2} = \frac{\lambda_3(1+r)r}{K_\lambda r^2 + br + K_\phi}, \qquad b = \phi_1\lambda_2 + \phi_2\lambda_1 - 2\lambda_3^2, \quad K_\phi = \phi_1\phi_2 - \lambda_3^2.$$

The numerator of its derivative w.r.t. r is $\lambda_3[(b - K_\lambda)r^2 + 2K_\phi r + K_\phi]$, where $b - K_\lambda = \beta_1\lambda_2 + \beta_2\lambda_1 + K_\lambda > 0$, hence $|a^T|$ is increasing in r. Next, the side coefficient of the market is $a^M = \frac{\lambda_3\gamma^M}{K_\lambda} = \frac{\lambda_3 r}{(1+r)K_\lambda}$, and

$$a^T - a^M = \frac{\lambda_3 r[(K_\lambda - K_\phi) + (2K_\lambda - b)r]}{(1+r)K_\lambda(K_\lambda r^2 + br + K_\phi)},$$

where $K_\lambda - K_\phi = \lambda_1\lambda_2 - \phi_1\phi_2 < 0$ and $2K_\lambda - b = -(\beta_1\lambda_2 + \beta_2\lambda_1) < 0$. Hence $a^T - a^M$ has the sign of $-\lambda_3$, while a^T has the sign of λ_3, so $|a^T| < |a^M| < \frac{|\lambda_3|}{K_\lambda}$, since $\gamma^M < 1$. The limit is obvious from the formula, hence proven

A.4.3 Proof of Theorem 1

Part (i). Here $\lambda_3 = 0$ and I use the formulas of Section 4.2.

Main weights. Obviously follows from the formula for the functions: β_3 is only in the denominator, and it is decreasing in $|\beta_3|$, hence the overall function is increasing and depends on $|\beta_3|$ (even)

Side weights. Obviously follows from the formula for the functions: when $\beta_3 \geq 0$, the numerator is increasing and the denominator is decreasing in β_3, hence the function is increasing. A change in sign of β_3 changes only the sign of the numerator, hence odd, and since the function is continuous, it is overall increasing.

Welfare.
$$\mathrm{WL}^M = \frac{(\lambda_1+\lambda_2)\tau_\epsilon}{2\lambda_1\lambda_2(\tau+\tau_\epsilon)^2} + \frac{(\phi_1+\phi_2)\tau}{2K_\phi(\tau+\tau_\epsilon)^2}; \mathrm{WL}^T = \frac{F_1+F_2}{2K_F}$$

The behavior of this function established in A.2 obviously follows from its form

As for $\mathrm{VI} = \frac{\phi_1+\phi_2}{2\tau_\theta K_\phi}$, the numerator does not depend on β_3, so $\mathrm{VI} \sim \frac{1}{K_\phi}$ for the comparative statics w.r.t. β_3

Now consider $\mathrm{WL}^T/\mathrm{VI} = \frac{\tau_\theta K_\phi(F_1+F_2)}{(\phi_1+\phi_2)K_F} \sim \frac{K_\phi}{K_F} = \frac{1}{\tau^2+\frac{C}{K_\phi}}$, where C is independent of β_3, hence it is an even function decreasing in $|\beta_3|$

And $\mathrm{WL}^M/\mathrm{VI} = \frac{\tau_\theta[K_\phi(\lambda_1+\lambda_2)\tau_\epsilon+\lambda_1\lambda_2(\phi_1+\phi_2)\tau]}{\lambda_1\lambda_2(\phi_1+\phi_2)(\tau+\tau_\epsilon)^2} \sim K_\phi(\lambda_1+\lambda_2)\tau_\epsilon$ (here we removed everything that is fixed w.r.t β_3), it is an even function decreasing in $|\beta_3|$, hence proven

Part (ii). Here $\beta_3 = 0$.

Main weights. In the case of $\beta_3 = 0$ we can find that

$$\gamma^T_{jj} = \frac{(\lambda_1\lambda_2-\lambda_3^2)(\phi_{-j}\tau+\lambda_{-j}\tau_\epsilon)\tau_\epsilon}{\lambda_{-j}[(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon)-\lambda_3^2(\tau+\tau_\epsilon)^2]}$$

Removing all unnecessary terms we are left with

$$\gamma^T_{jj} \sim \frac{\lambda_1\lambda_2-\lambda_3^2}{\frac{(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon)}{(\tau+\tau_\epsilon)^2}-\lambda_3^2}$$

Since $\frac{(\phi_j\tau+\lambda_j\tau_\epsilon)}{(\tau+\tau_\epsilon)} > \lambda_j$, we have $\frac{(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon)}{(\tau+\tau_\epsilon)^2} > \lambda_1\lambda_2$, hence this function is decreasing in $|\lambda_3|$

Side weights. In the case of $\beta_3 = 0$ we have $F_3 = \lambda_3(\tau+\tau_\epsilon)$ and $K_F = F_1F_2 - \lambda_3^2(\tau+\tau_\epsilon)^2$, where F_1, F_2 do not depend on λ_3. Hence the formula of Section 4.2 gives $\gamma^T_{12} = \gamma^T_{21} =$

$\frac{K_\lambda(\tau+\tau_\epsilon)\tau_\epsilon}{K_F}$, which is an even function of λ_3, and the side weights are

$$a^T = \frac{F_3\tau_\epsilon}{K_F} = \frac{\lambda_3(\tau+\tau_\epsilon)\tau_\epsilon}{F_1F_2 - \lambda_3^2(\tau+\tau_\epsilon)^2}.$$

When $\lambda_3 \geq 0$, the numerator is increasing and the denominator is decreasing in λ_3, hence the function is increasing. A change in sign of λ_3 changes only the sign of the numerator, hence odd, and since the function is continuous, it is overall increasing.

Welfare. In that case it can be easily established that (but now $K_\lambda = \lambda_1\lambda_2 - \lambda_3^2, K_\phi = \phi_1\phi_2 - \lambda_3^2$):

$$\mathrm{WL}^M = 0.5\left[\frac{(\lambda_1+\lambda_2)\tau_\epsilon}{K_\lambda(\tau+\tau_\epsilon)^2} + \frac{(\phi_1+\phi_2)\tau}{K_\phi(\tau+\tau_\epsilon)^2}\right]$$

$$\mathrm{WL}^T = 0.5\left[\frac{(\phi_1+\phi_2)\tau + (\lambda_1+\lambda_2)\tau_\epsilon}{(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon) - \lambda_3^2(\tau+\tau_\epsilon)^2}\right]$$

$$\mathrm{VI} = \frac{\phi_1+\phi_2}{2\tau_\theta K_\phi}$$

As for the behavior of $\mathrm{WL}^M, \mathrm{WL}^T$ everything is obvious from the formula for the functions.

$\mathrm{WL}^M/\mathrm{VI} \sim \frac{K_\phi}{K_\lambda} + C = \frac{\phi_1\phi_2-\lambda_3^2}{\lambda_1\lambda_2-\lambda_3^2} + C$, where C is independent of λ_3, and since $\phi_1\phi_2 > \lambda_1\lambda_2$ this function is increasing in $|\lambda_3|$

$$\mathrm{WL}^T/\mathrm{VI} \sim \frac{\phi_1\phi_2 - \lambda_3^2}{\frac{(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon)}{(\tau+\tau_\epsilon)^2} - \lambda_3^2}$$

And since $\frac{(\phi_j\tau+\lambda_j\tau_\epsilon)}{(\tau+\tau_\epsilon)} < \phi_j \Rightarrow \frac{(\phi_1\tau+\lambda_1\tau_\epsilon)(\phi_2\tau+\lambda_2\tau_\epsilon)}{(\tau+\tau_\epsilon)^2} < \phi_1\phi_2$ this function is increasing in $|\lambda_3|$

A.4.4 Proof of Proposition 4

i)From the formulas of Section 4.2 the coefficients of X_{i1}^T on s_{i1}, s_{i2} are $-\frac{\lambda_2\gamma_{11}^T}{K_\lambda} = -\frac{F_2\tau_\epsilon}{K_F}$ and $\frac{\lambda_3\gamma_{12}^T}{K_\lambda} = \frac{F_3\tau_\epsilon}{K_F}$, and the ones of X_{i2}^T are $\frac{F_3\tau_\epsilon}{K_F}$ and $-\frac{F_1\tau_\epsilon}{K_F}$, i.e. $C^T = \tau_\epsilon(\tau\Phi + \tau_\epsilon\Lambda)^{-1}$ as stated in

Section 4.2. Here F_1, F_2 do not contain β_3, λ_3, hence these coefficients depend on the linkages only through F_3. Then

$$\frac{\partial a^T}{\partial F_3} = \frac{(F_1F_2 + F_3^2)\tau_\epsilon}{K_F^2} > 0,$$

and a change in sign of F_3 changes only the sign of the numerator, hence odd. Since $\frac{\partial F_3}{\partial \beta_3} = \tau > 0$ and $\frac{\partial F_3}{\partial \lambda_3} = \tau + \tau_\epsilon > 0$, the side weight is increasing in both β_3 and λ_3. As for WL^T, by Subsection A.3 $\mathrm{WL}^T = \frac{\mathrm{tr}(C^T)}{2\tau_\epsilon} = \frac{F_1+F_2}{2(F_1F_2-F_3^2)}$, which depends on the linkages only through F_3 and is even and increasing in $|F_3|$.

ii)Obviously follows from the formula: $\gamma_{jj}^T = \frac{K_\lambda F_{-j}\tau_\epsilon}{\lambda_{-j}K_F} = \frac{K_\lambda}{\lambda_1\lambda_2} \cdot \frac{\lambda_j F_{-j}\tau_\epsilon}{F_1F_2-F_3^2}$, where λ_j, F_{-j} do not depend on the linkages and $F_1F_2-F_3^2$ is decreasing in $|F_3|$, hence g_j is even and increasing in $|F_3|$. For $\lambda_3 = 0$ we have $F_3 = \beta_3\tau$ and g_j is exactly the formula for γ_j^T of Section 5.2. The claim about $\frac{K_\lambda}{\lambda_1\lambda_2}$ is obvious, hence proven.

A.4.5 Joint parity of the exogenous solutions

Here I show that with exogenous public information and $\rho = 0$ the market equilibrium and the team solution are invariant under the joint sign change $(\beta_3, \lambda_3) \to (-\beta_3, -\lambda_3)$ together with the sign change of the coefficients on the signals about the other market, so that γ^T's, WL^T and WL^M are even and a^T's are odd under this change. Let $S = \mathrm{diag}(1, -1)$, so that $S = S^{-1} = S'$. The joint sign change replaces Λ, B, Φ by $S\Lambda S, SBS, S\Phi S$, and $(S\Lambda S)^{-1} = S\Lambda^{-1}S$. Let me use the representation of the welfare losses of Subsection A.3. For any coefficients C, the matrix SCS in the new economy comes with the matrix on public signals $S\Lambda^{-1}S - SCS = SDS$, and

$$(SCS)'(S\Lambda S)(SCS) = S(C'\Lambda C)S,$$

$$(SDS)'(S\Lambda S)(S\Phi^{-1}S)(S\Lambda S)(SDS) = S(D'\Lambda\Phi^{-1}\Lambda D)S,$$

while S does not change the diagonal entries. Hence the losses of SCS in the new economy are equal to the losses of C in the old one, the team solution of the new economy is SC^TS, which has the same diagonal and the opposite off-diagonal entries, and WL^T is the same. The market coefficients $C^M = \gamma^M\Lambda^{-1}$ are mapped in the same way, and WL^M does not change by its formula. Finally, $(\Lambda^{-1})_{jj}$ does not change while $(\Lambda^{-1})_{12}$ changes sign, so none of the γ's change, hence proven

A.4.6 Proof of Proposition 5

By Subsection A.3, $\mathrm{WL}^M = \frac{a\tau_\epsilon}{(\tau+\tau_\epsilon)^2} + \frac{b}{\tau+\tau_\epsilon}$ with $a = \frac{1}{2}\mathrm{tr}(\Lambda^{-1}) - \frac{1}{2}\mathrm{tr}(\Phi^{-1})$ and $b = \frac{1}{2}\mathrm{tr}(\Phi^{-1})$, and, as in the proof of Proposition 10, its derivative w.r.t. τ_ϵ is $\frac{(a-b)\tau-(a+b)\tau_\epsilon}{(\tau+\tau_\epsilon)^3}$. Hence WL^M is increasing in τ_ϵ for $\frac{\tau_\epsilon}{\tau} < \frac{a-b}{a+b} = 1 - \frac{2\mathrm{tr}(\Phi^{-1})}{\mathrm{tr}(\Lambda^{-1})} = 1 - \frac{2}{R}$ and decreasing after, and this interval is non-empty if and only if $R > 2$.

i)If $\lambda_3 = 0$, then $\mathrm{tr}(\Lambda^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ does not depend on β_3, while $\mathrm{tr}(\Phi^{-1}) = \frac{\phi_1+\phi_2}{\phi_1\phi_2-\beta_3^2}$ is even and increasing in $|\beta_3|$, hence R is even and decreasing in $|\beta_3|$.

ii)If $\beta_3 = 0$, then $R = \frac{\lambda_1+\lambda_2}{\phi_1+\phi_2} \cdot \frac{\phi_1\phi_2-\lambda_3^2}{\lambda_1\lambda_2-\lambda_3^2}$, and $\frac{A-x}{B-x}$ with $A > B > x$ is increasing in x, since its derivative is $\frac{A-B}{(B-x)^2}$. Here $A = \phi_1\phi_2 > B = \lambda_1\lambda_2 > x = \lambda_3^2$, hence R is even and increasing in $|\lambda_3|$.

iii)Obviously follows from $\Phi = (1+k)\Lambda$.

Finally, $\mathrm{VI} = \frac{\mathrm{tr}(\Phi^{-1})}{2\tau_\theta}$, and substituting $\mathrm{tr}(\Lambda^{-1}) = R\,\mathrm{tr}(\Phi^{-1})$ into WL^M gives the formula for $\frac{\mathrm{WL}^M}{\mathrm{VI}}$ in Section 5.2.

A.4.7 Proof of Theorem 2

Let $\beta_j = k\lambda_j$ for all $j \in \{1,2,3\}$, so that $\phi_j = (1+k)\lambda_j, j \in \{1,2,3\}$ and $\Phi = (1+k)\Lambda$. Let me denote $M = (1+k)\tau + \tau_\epsilon$ and notice that then

$$F_j = \phi_j\tau + \lambda_j\tau_\epsilon = \lambda_j M, j \in \{1,2,3\},$$

which is the whole content of the proof: the proportionality makes every entry of the problem carry the same scalar M. In the matrix form of Section 4.2 this means $\tau\Phi + \tau_\epsilon\Lambda = M\Lambda$, hence $C^T = \frac{\tau_\epsilon}{M}\Lambda^{-1}$ at once.

Plugging this into the formulas for the team solution of Section 4.2, we get

$$K_F = F_1F_2 - F_3^2 = M^2(\lambda_1\lambda_2 - \lambda_3^2) = M^2K_\lambda,$$

and hence

$$\gamma_{jj}^T = \frac{K_\lambda F_{-j}\tau_\epsilon}{\lambda_{-j}K_F} = \frac{K_\lambda\lambda_{-j}M\tau_\epsilon}{\lambda_{-j}M^2K_\lambda} = \frac{\tau_\epsilon}{M};\ \gamma_{12}^T = \gamma_{21}^T = \frac{K_\lambda F_3\tau_\epsilon}{\lambda_3K_F} = \frac{\tau_\epsilon}{M},$$

so that all four weights coincide and are equal to $\frac{\tau_\epsilon}{(1+k)\tau+\tau_\epsilon}$, which does not contain λ_3 at all. This gives the first part of (i), and also (ii), since $\gamma^M = \frac{\tau_\epsilon}{\tau+\tau_\epsilon}$ does not depend on the linkages either.

For the second part of (i), let us write the schedules of Section 4.2 in vector notation. With all four γ's equal to some γ, the coefficients of X_{i1}^T on (s_{i1}, s_{i2}) are $(-\frac{\lambda_2\gamma}{K_\lambda}, \frac{\lambda_3\gamma}{K_\lambda})$ and the ones of X_{i2}^T are $(\frac{\lambda_3\gamma}{K_\lambda}, -\frac{\lambda_1\gamma}{K_\lambda})$, i.e. the matrix of coefficients on private signals is

$$-\frac{\gamma}{K_\lambda}\begin{pmatrix}\lambda_2 & -\lambda_3\\ -\lambda_3 & \lambda_1\end{pmatrix} = -\gamma\Lambda^{-1},$$

and in the same way the one on public estimates is $-(1-\gamma)\Lambda^{-1}$, while the one on prices is

Λ^{-1} in any case. Hence under proportionality the team schedules have exactly the shape of the market ones, $x_i = \Lambda^{-1}(p - \gamma s_i - (1-\gamma)\hat{\theta})$, with a single scalar weight, hence (i) is proven. It is instructive to see the value of this weight once again from the welfare side, because it also gives (iii). Then $x_i - \tilde{x} = -\gamma\Lambda^{-1}\varepsilon_i$ and, from market clearing, $\tilde{x} - x^0 = (1-\gamma)\Phi^{-1}(\theta - \hat{\theta})$, so that with $\rho = 0$

$$\mathrm{WL}(\gamma) = \frac{\gamma^2}{2}\frac{tr(\Lambda^{-1})}{\tau_\epsilon} + \frac{(1-\gamma)^2}{2}\frac{tr(\Phi^{-1})}{\tau} = \frac{tr(\Lambda^{-1})}{2}\left[\frac{\gamma^2}{\tau_\epsilon} + \frac{(1-\gamma)^2}{(1+k)\tau}\right],$$

where I used $tr(\Phi^{-1}) = \frac{tr(\Lambda^{-1})}{1+k}$. Minimizing the bracket over γ gives $\frac{\gamma}{\tau_\epsilon} = \frac{1-\gamma}{(1+k)\tau}$, i.e. $\gamma^T = \frac{\tau_\epsilon}{(1+k)\tau+\tau_\epsilon}$ once again, while for the market equilibrium the same formula holds with $\gamma = \gamma^M = \frac{\tau_\epsilon}{\tau+\tau_\epsilon}$.

Finally, since $\Phi^{-1} = \frac{\Lambda^{-1}}{1+k}$,

$$\mathrm{VI} = \frac{\mathrm{tr}(\Phi^{-1})}{2\tau_\theta} = \frac{\lambda_1 + \lambda_2}{2\tau_\theta(1+k)K_\lambda},$$

and hence for both solutions

$$\frac{\mathrm{WL}}{\mathrm{VI}} = \tau_\theta\left[(1+k)\frac{\gamma^2}{\tau_\epsilon} + \frac{(1-\gamma)^2}{\tau}\right],$$

in which K_λ has cancelled and no linkage parameter is left. At the same time $tr(\Lambda^{-1}) = \frac{\lambda_1+\lambda_2}{K_\lambda}$ is increasing in $|\lambda_3|$ and tends to infinity as $|\lambda_3| \to \sqrt{\lambda_1\lambda_2}$, while the bracket in $\mathrm{WL}(\gamma)$ does not depend on λ_3, so that $\mathrm{WL}^T, \mathrm{WL}^M$ themselves are increasing in $|\lambda_3|$ and unbounded along the manifold, hence (iii) is proven.

Two remarks about what is used here. First, $\rho = 0$ is necessary: with $\rho \neq 0$ the loss acquires the cross terms of Subsection A.1 and the scalar reduction above breaks. For symmetric markets the formulas of Section 6.1 with $\phi_\pm = (1+k)\lambda_\pm$ give $\gamma_{11}^T = \frac{\tau_\epsilon}{2\lambda}\left(\frac{\lambda-\lambda_3}{m_+} + \frac{\lambda+\lambda_3}{m_-}\right)$ with $m_\pm = (1+k)\tau_\pm + \tau_\epsilon$, which is linear in λ_3 with a slope of the same sign as $-\rho$, since $m_+ < m_-$ for $\rho > 0$ and $m_+ > m_-$ for $\rho < 0$. Second, proportionality is needed for the

whole matrix: if $\frac{\beta_1}{\lambda_1} \neq \frac{\beta_2}{\lambda_2}$ then $\phi_j\tau + \lambda_j\tau_\epsilon$ is no longer λ_j times a common scalar, so $\tau\Phi + \tau_\epsilon\Lambda$ is not proportional to Λ, and the scalar reduction above breaks.

Finally, let me show what remains without the proportionality of the whole matrices. By the formulas of Section 4.2, $\gamma^T_{jj} = \frac{K_\lambda F_{-j}\tau_\epsilon}{\lambda_{-j}K_F}$ is equal to its value for independent markets, $\frac{\lambda_j\tau_\epsilon}{F_j}$, if and only if $K_\lambda F_1 F_2 = \lambda_1\lambda_2 K_F$, i.e. $\lambda_1\lambda_2 F_3^2 = \lambda_3^2 F_1 F_2$, which is one and the same condition for $j = 1$ and $j = 2$. Let $k_j = \frac{\beta_j}{\lambda_j}$, so that $F_j = \lambda_j((1 + k_j)\tau + \tau_\epsilon)$, $j \in \{1, 2\}$. Then along the ray $\beta_3 = \bar{k}\lambda_3$, where

$$(1 + \bar{k})\tau + \tau_\epsilon = \sqrt{((1 + k_1)\tau + \tau_\epsilon)((1 + k_2)\tau + \tau_\epsilon)},$$

we have $F_3 = \lambda_3((1 + \bar{k})\tau + \tau_\epsilon)$, and the condition holds, so both main weights stay at their values for independent markets. For $k_1 = k_2 = k$ this gives $\bar{k} = k$, i.e. the proportionality manifold. For $k_1 \neq k_2$ the ray depends on $\frac{\tau_\epsilon}{\tau}$, and along it $\tau\Phi + \tau_\epsilon\Lambda$ is not proportional to Λ, so the team solution does not take the form of the market one.

A.4.8 Team solution for symmetric markets

Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_\pm = \frac{1}{2}(I \pm J)$, so that $P_+ + P_- = I$, $P_\pm^2 = P_\pm$, $P_+P_- = 0$ and $\mathrm{tr}P_\pm = 1$. With symmetric markets $\Lambda = \lambda_+P_+ + \lambda_-P_-$, $\Phi = \phi_+P_+ + \phi_-P_-$, the covariance matrix of the public errors is $\Sigma_e = \tau^{-1}(I + \rho J) = \tau_+^{-1}P_+ + \tau_-^{-1}P_-$, and $\Sigma_\varepsilon = \tau_\epsilon^{-1}I$. For $C = c_+P_+ + c_-P_-$ we have $D = \Lambda^{-1} - C = (\lambda_+^{-1} - c_+)P_+ + (\lambda_-^{-1} - c_-)P_-$, and the representation of Subsection A.3 gives

$$\mathrm{WL} = \frac{1}{2}\sum_{s=\pm}\left(\frac{c_s^2\lambda_s}{\tau_\epsilon} + \frac{(\lambda_s^{-1} - c_s)^2\lambda_s^2}{\phi_s\tau_s}\right),$$

which is the sum of two losses of the form of V17. Each of them is minimized at $c_s = \frac{\tau_\epsilon}{\tau_s\phi_s + \tau_\epsilon\lambda_s} = \frac{\tau_\epsilon}{H_s}$, and the resulting C satisfies the F.O.C. $\Lambda C\Sigma_\varepsilon = \Lambda\Phi^{-1}\Lambda D\Sigma_e$ of Subsection A.3, since all matrices here are combinations of P_+, P_-; as that F.O.C. has a unique solution, this is the team solution. Then by Subsection A.3 $\mathrm{WL}^T = \frac{\mathrm{tr}(C^T)}{2\tau_\epsilon} = \frac{1}{2}\left(\frac{1}{H_+} + \frac{1}{H_-}\right)$. The main

coefficient is $(C^T)_{11} = \frac{1}{2}(c_+ + c_-)$ and $(\Lambda^{-1})_{11} = \frac{1}{2}(\lambda_+^{-1} + \lambda_-^{-1}) = \frac{\lambda}{K_\lambda}$, so $\gamma_{11}^T = \frac{K_\lambda}{\lambda}(C^T)_{11} = \frac{K_\lambda \tau_\epsilon}{\lambda}\mathrm{WL}^T$, and the side weight is $a^T = -(C^T)_{12} = \frac{1}{2}(c_- - c_+) = \frac{\tau_\epsilon(H_+ - H_-)}{2H_+H_-}$. Finally, with $\phi_3 = \beta_3 + \lambda_3$,

$$H_+ - H_- = \tau \frac{(\phi + \phi_3)(1-\rho) - (\phi - \phi_3)(1+\rho)}{1-\rho^2} + 2\tau_\epsilon \lambda_3,$$

$$H_+ - H_- = \frac{2[\phi_3 \tau - \rho\phi\tau + (1-\rho^2)\lambda_3\tau_\epsilon]}{1-\rho^2} = \frac{2G}{1-\rho^2},$$

hence proven.

For $\lambda_3 = 0$ we have $(1 \mp \rho)H_\mp = F \mp \beta_3\tau \mp \lambda\tau_\epsilon\rho$ with $F = \phi\tau + \lambda\tau_\epsilon$, which gives the formulas of Subsection A.1, and for $\beta_3 = 0$ we have $(1 \mp \rho)H_\mp = (\phi \mp \lambda_3)\tau + (\lambda \mp \lambda_3)(1 \mp \rho)\tau_\epsilon$.

A.4.9 Proof of Proposition 6

By Appendix A.4.8, $\mathrm{WL}^T = \frac{1}{2}\left(\frac{1}{H_+} + \frac{1}{H_-}\right)$ and $\gamma_{jj}^T = \frac{K_\lambda \tau_\epsilon}{\lambda}\mathrm{WL}^T$, where the factor $\frac{K_\lambda \tau_\epsilon}{\lambda}$ depends neither on ρ nor on β_3.

i)Let us notice that $\frac{1}{H_\pm} = \frac{1 \pm \rho}{\tau\phi_\pm + \tau_\epsilon\lambda_\pm(1 \pm \rho)}$, and the function $\frac{x}{A+Bx}$ with $A, B > 0$ is increasing and concave in $x > 0$, since its second derivative is $-\frac{2AB}{(A+Bx)^3}$. Since $1 \pm \rho$ is linear in ρ, both terms are concave in ρ, hence WL^T and γ^T's are concave in ρ.

ii)$H_\pm$ are positive and linear in β_3, hence $\frac{1}{H_\pm}$ are convex in β_3, and so are WL^T and γ^T's.

iii)In the same way $H_\pm$ are positive and linear in λ_3, hence WL^T is convex in λ_3. As for γ^T's, $\gamma_{11}^T = \frac{\tau_\epsilon}{2\lambda}\left(\frac{\lambda_+\lambda_-}{H_+} + \frac{\lambda_+\lambda_-}{H_-}\right)$, and let me write $H_+ = A_+ + k_+\lambda_+$ with $A_+ = \tau_+(\beta + \beta_3) \geq 0$ and $k_+ = \tau_+ + \tau_\epsilon > 0$. Then $\frac{\lambda_+\lambda_-}{H_+} = fg$ with $f = \frac{\lambda_+}{A_+ + k_+\lambda_+}$, which is positive, increasing and concave in λ_3, and $g = \lambda_-$, which is positive and linearly decreasing in λ_3, so that $(fg)'' = f''g + 2f'g' \leq 0$. The same argument works for the second term with $H_- = A_- + k_-\lambda_-$, $A_- = \tau_-(\beta - \beta_3) \geq 0$, $k_- = \tau_- + \tau_\epsilon$ and the roles of λ_+, λ_- exchanged. Hence γ^T's are concave in λ_3, hence proven.

Concave (convex) functions are quasi-concave (quasi-convex), and their peak (dip) can be located at the boundary of the domain, which gives the last claim.

A.4.10 Proof of Proposition 7

i)By Appendix A.4.8, $a^T = \frac{\tau_\epsilon}{2}\left(\frac{1}{H_-} - \frac{1}{H_+}\right)$. Here $\frac{\partial H_+}{\partial \beta_3} = \tau_+ > 0$, $\frac{\partial H_-}{\partial \beta_3} = -\tau_- < 0$, $\frac{\partial H_+}{\partial \lambda_3} = \tau_+ + \tau_\epsilon > 0$ and $\frac{\partial H_-}{\partial \lambda_3} = -(\tau_- + \tau_\epsilon) < 0$, hence a^T is increasing in β_3 and λ_3. Next, $H_\pm = \frac{\tau\phi_\pm}{1\pm\rho} + \tau_\epsilon\lambda_\pm$, so $\frac{\partial H_+}{\partial \rho} = -\frac{\tau\phi_+}{(1+\rho)^2} < 0$ and $\frac{\partial H_-}{\partial \rho} = \frac{\tau\phi_-}{(1-\rho)^2} > 0$, since $\phi_\pm > 0$ as the eigenvalues of Φ, hence a^T is decreasing in ρ. The sign of a^T is the sign of $H_+ - H_- = \frac{2G}{1-\rho^2}$, i.e. of G. Since a^T is monotone in each of β_3, λ_3, ρ, it changes sign at most once along each of them, so $|a^T|$ is first decreasing and then increasing, i.e. quasi-convex.

ii)Obviously follows from $G = 0$: for $\lambda_3 = 0$ it reads $\beta_3 = \rho\phi$, which is admissible whenever $|\rho|\phi \le \beta$; for $\beta_3 = 0$ it reads $\lambda_3(\tau + (1-\rho^2)\tau_\epsilon) = \rho\phi\tau$; and for $\rho = 0$ it reads $\beta_3\tau + \lambda_3(\tau + \tau_\epsilon) = 0$, hence proven

A.4.11 Proof of Proposition 8

Let $r = \frac{\tau_\epsilon}{\tau}$ and $b_\pm = \frac{\phi_\pm}{1\pm\rho} > 0$, so that $H_\pm = \tau(b_\pm + r\lambda_\pm)$.

i)By Appendix A.4.8, $\gamma_{11}^T = \frac{K_\lambda \tau_\epsilon}{2\lambda}\left(\frac{1}{H_+} + \frac{1}{H_-}\right) = \frac{K_\lambda}{2\lambda}\left(\frac{r}{b_+ + r\lambda_+} + \frac{r}{b_- + r\lambda_-}\right)$, and both terms are increasing in r, hence proven.

ii)Obviously follows from the formula $\mathrm{WL}^T = \frac{1}{2}\left(\frac{1}{H_+} + \frac{1}{H_-}\right)$, since $H_\pm$ are increasing in τ and τ_ϵ.

iii)$a^T = \frac{\tau_\epsilon}{2}\left(\frac{1}{H_-} - \frac{1}{H_+}\right) = \frac{1}{2}\left(\frac{r}{b_- + r\lambda_-} - \frac{r}{b_+ + r\lambda_+}\right)$ depends only on r, tends to 0 as $r \to 0$ and to $\frac{1}{2}\left(\frac{1}{\lambda_-} - \frac{1}{\lambda_+}\right) = \frac{\lambda_3}{K_\lambda}$ as $r \to \infty$. Its derivative is $\frac{1}{2}\left(\frac{b_-}{(b_- + r\lambda_-)^2} - \frac{b_+}{(b_+ + r\lambda_+)^2}\right)$, and it is equal to zero if and only if $\sqrt{b_-}(b_+ + r\lambda_+) = \sqrt{b_+}(b_- + r\lambda_-)$, which is a linear equation in r and hence has at most one root, unless $a^T \equiv 0$. If $\lambda_3 = 0$, so that $\lambda_\pm = \lambda$, and $\beta_3 \neq \rho\phi$,

so that $b_+ \neq b_-$, then a^T tends to 0 at both ends and is not identically zero, hence its only critical point $r^* = \frac{\sqrt{b_+ b_-}}{\lambda}$ is its extremum and $|a^T|$ is quasi-concave in r; since $b_+ b_- = \frac{\phi^2 - \beta_3^2}{1-\rho^2}$, this gives $\frac{\tau}{\tau_\epsilon} = \lambda\sqrt{\frac{1-\rho^2}{\phi^2-\beta_3^2}}$. Finally, a function of $\frac{\tau_\epsilon}{\tau}$ that is quasi-concave in it is quasi-concave in (τ, τ_ϵ), since its upper level sets are cones between two rays, hence proven.

A.5 Market equilibrium with endogenous information

First of all we obtain public signals, jointly informationally equivalent to p_1, p_2. This could be done from market-clearing conditions that give us the functions $p_k(a_{ij}, \theta_j, u_j)$ with some linear transformations, separating θ_1, θ_2

$$z_1 = K^*\theta_1 + (\beta_2 a_{22} - \beta_3 a_{12})u_1 - (\beta_3 a_{22} - \beta_1 a_{12})u_2$$
$$z_2 = K^*\theta_2 + (\beta_1 a_{11} - \beta_3 a_{21})u_2 - (\beta_3 a_{11} - \beta_2 a_{21})u_1$$

where $K^* = K_\beta(a_{11}a_{22} - a_{12}a_{21})$.

And then to solve for the market equilibrium here we need two simple steps.

1)Construct the best linear estimator $\mathbb{E}[\theta_1|I_i] = as_{i1} + bs_{i2} + cz_1 + dz_2$.

Since $\mathbb{E}[(z_1 - K^*\theta_1)^2] = \frac{M_1}{\tau_u}$, $\mathbb{E}[(z_2 - K^*\theta_2)^2] = \frac{M_2}{\tau_u}$, $\mathbb{E}[(z_1 - K^*\theta_1, z_2 - K^*\theta_2)] = -\frac{M_3}{\tau_u}$, the coefficients in this estimator should solve the program:

$$\min_{a,b,c,d} \frac{(1 - a - cK^*)^2 + (b + dK^*)^2}{\tau_\theta} + \frac{a^2 + b^2}{\tau_\epsilon} + \frac{M_1 c^2 + M_2 d^2 - 2M_3 cd}{\tau_u}$$

2)Equate the coefficients a, b with $\lambda_1 a_{11}, -\lambda_1 a_{12}$.

Finally we need to repeat the symmetric operation for θ_2, but since everything is symmetric, we will just change the corresponding indices in the equations to obtain the other two equations.

For the market equilibrium I obtained that $a_{11}, a_{22}, a_{12}, a_{21}$ solve the following system

(this is not the exact system that will be obtained after doing the procedure above, but it could be easily derived via a couple of substitutions)

$$a_{11}\tau_\epsilon = \lambda_2(\tau_\theta + \tau_\epsilon + M_2\tau_u)(a_{11}a_{22} - a_{12}a_{21})$$

$$a_{22}\tau_\epsilon = \lambda_1(\tau_\theta + \tau_\epsilon + M_1\tau_u)(a_{11}a_{22} - a_{12}a_{21})$$

$$\lambda_1 a_{12}\tau_\epsilon = \lambda_2 a_{21}\tau_\epsilon = \lambda_1\lambda_2 M_3\tau_u(a_{11}a_{22} - a_{12}a_{21})$$

$$M_1 = (\beta_1 a_{11} - \beta_3 a_{21})^2 + (\beta_2 a_{21} - \beta_3 a_{11})^2$$

$$M_2 = (\beta_2 a_{22} - \beta_3 a_{12})^2 + (\beta_1 a_{12} - \beta_3 a_{22})^2$$

$$M_3 = (\beta_2 a_{22} - \beta_3 a_{12})(\beta_3 a_{11} - \beta_2 a_{21}) + (\beta_3 a_{22} - \beta_1 a_{12})(\beta_1 a_{11} - \beta_3 a_{21})$$

With $\lambda_3 \neq 0$ we need to modify the first three equations to:

$$a_{11}\tau_\epsilon = [\lambda_2(\tau_\theta + \tau_\epsilon + M_2\tau_u) + \lambda_3 M_3\tau_u](a_{11}a_{22} - a_{12}a_{21})$$

$$a_{22}\tau_\epsilon = [\lambda_1(\tau_\theta + \tau_\epsilon + M_1\tau_u) + \lambda_3 M_3\tau_u](a_{11}a_{22} - a_{12}a_{21})$$

$$a_{12}\tau_\epsilon = [\lambda_2 M_3\tau_u + \lambda_3(\tau_\theta + \tau_\epsilon + M_1\tau_u)](a_{11}a_{22} - a_{12}a_{21})$$

$$a_{21}\tau_\epsilon = [\lambda_1 M_3\tau_u + \lambda_3(\tau_\theta + \tau_\epsilon + M_2\tau_u)](a_{11}a_{22} - a_{12}a_{21})$$

A.5.1 Joint parity of the market equilibrium

Here I prove the claim for the market equilibrium; for the team solution see Subsection A.6. Let us apply to a solution the transformation

$$(\beta_3, \lambda_3) \to (-\beta_3, -\lambda_3),\ (a_{21}, a_{12}) \to (-a_{21}, -a_{12}),\ a_{11}, a_{22} \text{ unchanged}$$

and check the system term by term. First, $\Delta = a_{11}a_{22} - a_{12}a_{21}$ is unchanged, since both side coefficients change sign at once. Next, in M_1 the first bracket $\beta_1 a_{11} - \beta_3 a_{21}$ is unchanged, because β_3 and a_{21} change sign simultaneously, while the second bracket $\beta_2 a_{21} - \beta_3 a_{11}$ changes

sign and enters squared; hence M_1 is unchanged, and by the same argument so is M_2. As for M_3, in each of its two products exactly one factor changes sign, hence $M_3 \to -M_3$.

Now the first two equations are unchanged, since $\lambda_3 M_3 \to (-\lambda_3)(-M_3) = \lambda_3 M_3$ and everything else in them is invariant. In the last two equations both sides change sign: on the left because of a_{12}, a_{21}, on the right because $\lambda_2 M_3 \to -\lambda_2 M_3$ and $\lambda_3(\tau_\theta + \tau_\epsilon + M_1\tau_u) \to -\lambda_3(\tau_\theta + \tau_\epsilon + M_1\tau_u)$. So the transformed quadruplet solves the transformed system, and the claimed parity follows.

Let me stress that the separate parities are weaker than they look. If $\lambda_3 = 0$, the transformation above involves only β_3 and gives evenness of a_{11}, a_{22} and oddness of a_{21}, a_{12} in β_3; if $\beta_3 = 0$, it involves only λ_3 and gives the same in λ_3. But if both linkages are non-zero, neither of the two separate transformations works: changing the sign of a_{21} without changing the sign of β_3 turns $\beta_1 a_{11} - \beta_3 a_{21}$ into $\beta_1 a_{11} + \beta_3 a_{21}$ and M_1 is not invariant any more. So in the general model only the joint sign change is a symmetry, and I have checked numerically that the separate parities indeed fail there.

A.5.2 Side coefficients of the market

The system also implies an identity for the side coefficients of the market, which is used in Section 7.4. Let me multiply the third equation of the general system above by λ_1 and the fourth one by λ_2 and subtract, which gives

$$(\lambda_1 a_{12} - \lambda_2 a_{21})\tau_\epsilon = \lambda_3[\lambda_1(\tau_\theta + \tau_\epsilon + M_1\tau_u) - \lambda_2(\tau_\theta + \tau_\epsilon + M_2\tau_u)](a_{11}a_{22} - a_{12}a_{21}),$$

while subtracting the first equation from the second one gives

$$(a_{22} - a_{11})\tau_\epsilon = [\lambda_1(\tau_\theta + \tau_\epsilon + M_1\tau_u) - \lambda_2(\tau_\theta + \tau_\epsilon + M_2\tau_u)](a_{11}a_{22} - a_{12}a_{21}).$$

Hence $\lambda_1 a_{12} - \lambda_2 a_{21} = \lambda_3(a_{22} - a_{11})$. In matrix form the system says that $\tau_\epsilon (C^M)^{-1} = Q\Lambda$ with the symmetric matrix $Q = \begin{pmatrix} \tau_\theta + \tau_\epsilon + M_1\tau_u & M_3\tau_u \\ M_3\tau_u & \tau_\theta + \tau_\epsilon + M_2\tau_u \end{pmatrix}$ and $C^M = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$, so that $\Lambda C^M = \tau_\epsilon Q^{-1}$ is symmetric, which is the same identity: Q is the posterior precision of the estimates of θ_1, θ_2, and $\tau_\epsilon Q^{-1}$ is the matrix of the Bayesian weights on the private signals, exactly as $\Lambda C^M = \gamma^M I$ in the exogenous case. For $\lambda_3 = 0$ it gives $\lambda_1 a_{12} = \lambda_2 a_{21}$, i.e. the ratio $\frac{a_{12}}{a_{21}} = \frac{\lambda_2}{\lambda_1}$ is a constant of the model, and every statement about one of the side coefficients is automatically a statement about the other.

A.6 Team solution with endogenous information

First of all let us consider WLOG the following market schedules

$$X_{i1} = b_1 - a_{11}s_{i1} + a_{12}s_{i2} + c_1\hat{\theta}_1 + d_1\hat{\theta}_2 + e_1 p_1 + f_1 p_2$$
$$X_{i2} = b_2 - a_{22}s_{i2} + a_{21}s_{i1} + c_2\hat{\theta}_2 + d_2\hat{\theta}_1 + e_2 p_2 + f_2 p_1$$

Here we should use the same strategy of deriving b_j, c_j, d_j, e_j, f_j as functions of the a_{jk}'s as in the exogenous case in Subsection A.1, but now we must take into consideration that $\hat{\theta}_j \not\perp p_k$ and $\theta_j \not\perp p_k$. However, the BLP property $\mathbb{E}[(\theta_j - \hat{\theta}_j)\hat{\theta}_k] = 0, j, k \in \{1, 2\}$ still holds.

All this comes from the fact that prices are directly related to the θ's and the public signals z_1, z_2, which were introduced in the previous section and which are used in $\hat{\theta}$.

Currently I skip the enormous algebra that appears via analysis of the F.O.C.s due to endogenous correlation in public signals; however, I assure the reader that it is possible to verify that overall we come to the same functional form as in the exogenous case.

$$X_{ij}^T(I_i, p) = \lambda_j^{-1}(p_j - \gamma_j^T s_{ij} - (1 - \gamma_j^T)\hat{\theta}_j) + a_j^T(s_{i(-j)} - \hat{\theta}_{-j}), j \in \{1, 2\}$$

To deal with two different functional representations of schedules here we use the important relationship and a slight abuse of notation $a_{11}^T = \lambda_1^{-1}\gamma_1^T, a_{22}^T = \lambda_2^{-1}\gamma_2^T, a_1^T = a_{12}^T, a_2^T = a_{21}^T$.

And hence we need to recalculate only $\hat{\theta}_j, \mathbb{E}[(\hat{\theta}_j - \theta_j)^2] = \tau_j^{-1}, j \in \{1,2\}, \mathbb{E}[(\hat{\theta}_1 - \theta_1)(\hat{\theta}_2 - \theta_2)] = \tau_3^{-1}$ and then we would get the equation for the welfare losses for market equilibria.

Here we define M_1, M_2, M_3, K^* as in the previous subsection of the appendix.

Estimators for public information have the following form

$$\hat{\theta}_1 = \frac{\tau_u[(K^{*2}\tau_u + M_1\tau_\theta)z_1 + M_3\tau_\theta z_2]}{K^*(K^{*2}\tau_u^2 + (M_1+M_2)\tau_u\tau_\theta + \tau_\theta^2)} = \frac{z_1(1-\frac{\tau_\theta}{\tau_1}) + \frac{z_2\tau_\theta}{\tau_3}}{K^*}$$
$$\hat{\theta}_2 = \frac{\tau_u[(K^{*2}\tau_u + M_2\tau_\theta)z_2 + M_3\tau_\theta z_1]}{K^*(K^{*2}\tau_u^2 + (M_1+M_2)\tau_u\tau_\theta + \tau_\theta^2)} = \frac{z_2(1-\frac{\tau_\theta}{\tau_2}) + \frac{z_1\tau_\theta}{\tau_3}}{K^*}$$

Their variances and covariance have the following form:

$$\tau_1 = \frac{K^{*2}\tau_u^2 + (M_1+M_2)\tau_u\tau_\theta + \tau_\theta^2}{M_2\tau_u + \tau_\theta} = \tau_\theta + \frac{K^{*2}\tau_u + M_1\tau_\theta}{M_2\tau_u + \tau_\theta}\tau_u$$
$$\tau_2 = \frac{K^{*2}\tau_u^2 + (M_1+M_2)\tau_u\tau_\theta + \tau_\theta^2}{M_1\tau_u + \tau_\theta} = \tau_\theta + \frac{K^{*2}\tau_u + M_2\tau_\theta}{M_1\tau_u + \tau_\theta}\tau_u$$

$$\tau_3 = \frac{K^{*2}\tau_u^2 + (M_1+M_2)\tau_u\tau_\theta + \tau_\theta^2}{M_3\tau_u}$$

Finally if we plug all these into the welfare losses we would get:

$$\mathrm{WL}^T(\gamma_1,\gamma_2,a_{21},a_{12}) = \frac{\gamma_1^2}{2\lambda_1\tau_\epsilon} + \frac{\gamma_2^2}{2\lambda_2\tau_\epsilon} + \frac{\lambda_2 a_{21}^2}{2\tau_\epsilon} + \frac{\lambda_1 a_{12}^2}{2\tau_\epsilon}$$

$$+\frac{\phi_2}{2K_\phi\tau_1}(1-\gamma_1)^2 - \frac{\beta_3\lambda_2}{K_\phi\tau_1}(1-\gamma_1)a_{21} + \frac{\lambda_2^2\phi_1}{2K_\phi\tau_1}a_{21}^2 + \frac{\phi_1}{2K_\phi\tau_2}(1-\gamma_2)^2 - \frac{\beta_3\lambda_1}{K_\phi\tau_2}(1-\gamma_2)a_{12} + \frac{\lambda_1^2\phi_2}{2K_\phi\tau_2}a_{12}^2$$
$$+\frac{\beta_3}{K_\phi\tau_3}(1-\gamma_1)(1-\gamma_2) + \frac{\beta_3\lambda_1\lambda_2}{K_\phi\tau_3}a_{21}a_{12} - \frac{\lambda_1\phi_2}{K_\phi\tau_3}(1-\gamma_1)a_{12} - \frac{\lambda_2\phi_1}{K_\phi\tau_3}(1-\gamma_2)a_{21}$$

Unfortunately, even the equations jointly determining the optimal team solution are enormous, so I do not include them here, relying on numerical optimization.

As for arbitrary λ_3 (here $K_\lambda = \lambda_1\lambda_2 - \lambda_3^2$) the welfare losses have the following form:

$$\mathrm{WL}^T(a_{11}, a_{22}, a_{21}, a_{12}) = \frac{\lambda_1 a_{11}^2}{2\tau_{\epsilon 1}} - \frac{\lambda_3 a_{11} a_{21}}{\tau_{\epsilon 1}} + \frac{\lambda_2 a_{21}^2}{2\tau_{\epsilon 1}} + \frac{\lambda_2 a_{22}^2}{2\tau_{\epsilon 2}} - \frac{\lambda_3 a_{22} a_{12}}{\tau_{\epsilon 2}} + \frac{\lambda_1 a_{12}^2}{2\tau_{\epsilon 2}} -$$

$$-\frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau_1'}\left(\frac{\lambda_2}{K_\lambda} - a_{11}\right)\left(\frac{\lambda_3}{K_\lambda} - a_{21}\right) - \frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau_2'}\left(\frac{\lambda_1}{K_\lambda} - a_{22}\right)\left(\frac{\lambda_3}{K_\lambda} - a_{12}\right)$$

$$+\frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{2K_\phi \tau_1'}\left(\frac{\lambda_2}{K_\lambda} - a_{11}\right)^2 + \frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{2K_\phi \tau_1'}\left(\frac{\lambda_3}{K_\lambda} - a_{21}\right)^2$$

$$+\frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{2K_\phi \tau_2'}\left(\frac{\lambda_1}{K_\lambda} - a_{22}\right)^2 + \frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{2K_\phi \tau_2'}\left(\frac{\lambda_3}{K_\lambda} - a_{12}\right)^2$$

$$+\frac{\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda}{K_\phi \tau_3'}\left(\frac{\lambda_2}{K_\lambda} - a_{11}\right)\left(\frac{\lambda_3}{K_\lambda} - a_{12}\right) + \frac{\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda}{K_\phi \tau_3'}\left(\frac{\lambda_1}{K_\lambda} - a_{22}\right)\left(\frac{\lambda_3}{K_\lambda} - a_{21}\right) -$$

$$-\frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau_3'}\left(\frac{\lambda_2}{K_\lambda} - a_{11}\right)\left(\frac{\lambda_1}{K_\lambda} - a_{22}\right) - \frac{\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda}{K_\phi \tau_3'}\left(\frac{\lambda_3}{K_\lambda} - a_{21}\right)\left(\frac{\lambda_3}{K_\lambda} - a_{12}\right)$$

A.6.1 Joint parity of the team solution

Let us apply the joint sign change $(\beta_3, \lambda_3) \to (-\beta_3, -\lambda_3)$, $(a_{21}, a_{12}) \to (-a_{21}, -a_{12})$ to the loss function for arbitrary λ_3 above. K_λ, K_β are unchanged, and so is K_ϕ, since $\phi_3 = \beta_3 + \lambda_3$ only changes sign; hence $\lambda_1(K_\phi - K_\beta) - \beta_1 K_\lambda$ and $\lambda_2(K_\phi - K_\beta) - \beta_2 K_\lambda$ are unchanged, while $\lambda_3(K_\phi - K_\beta) - \beta_3 K_\lambda$ changes sign. The brackets $\frac{\lambda_2}{K_\lambda} - a_{11}, \frac{\lambda_1}{K_\lambda} - a_{22}$ are unchanged, while $\frac{\lambda_3}{K_\lambda} - a_{21}, \frac{\lambda_3}{K_\lambda} - a_{12}$ change sign. Finally, τ_1', τ_2' are unchanged and τ_3' changes sign, since they are built from K^*, M_1, M_2, M_3 in the same way as τ_1, τ_2, τ_3, and by the proof in Subsection A.5 K^*, M_1, M_2 are unchanged and M_3 changes sign. Now it is a matter of direct inspection that every term of the loss contains an even number (zero, two or four) of the factors that change sign, so the loss itself is unchanged. Hence if $(a_{11}, a_{22}, a_{21}, a_{12})$ minimizes it for (β_3, λ_3), then $(a_{11}, a_{22}, -a_{21}, -a_{12})$ minimizes it for $(-\beta_3, -\lambda_3)$, hence proven

A.6.2 Proof of Proposition 9

Holding γ, a's fixed, WL^T is decreasing in τ_ϵ. And since γ^T, a^T's minimize WL^T holding other parameters fixed, by the envelope theorem the statement of the proposition is true for τ_ϵ.

For τ_θ and τ_u the envelope part of the argument is exactly the same, but the first step needs more care, because these two parameters enter WL^T only through τ_1, τ_2, τ_3, and the terms with τ_3 can have either sign, so it is not enough to look at them one by one.

Let us hold all a's fixed, denote by $e = (\theta_1 - \hat{\theta}_1, \theta_2 - \hat{\theta}_2)'$ the vector of public information errors and by Σ its covariance matrix, so that by the definitions above

$$\Sigma = \begin{pmatrix} \tau_1^{-1} & \tau_3^{-1} \\ \tau_3^{-1} & \tau_2^{-1} \end{pmatrix}.$$

Recall that $z = K^*\theta + \nu$, where ν has the covariance matrix $V = \tau_u^{-1} \begin{pmatrix} M_1 & -M_3 \\ -M_3 & M_2 \end{pmatrix}$, and that K^*, M_1, M_2, M_3 depend only on the a's and on the demand parameters, and *not* on τ_θ or τ_u. Notice also that V is positive semidefinite precisely because $M_1 M_2 - M_3^2 = K^{*2} \geq 0$, which is the same fact that gives $\tau_3^2 \geq \tau_1 \tau_2$. Then from the standard Gaussian formula we get

$$\Sigma = \left(\tau_\theta I + K^{*2} \tau_u \begin{pmatrix} M_1 & -M_3 \\ -M_3 & M_2 \end{pmatrix}^{-1} \right)^{-1},$$

and it is immediate that increasing either τ_θ or τ_u increases the matrix in the brackets in the positive semidefinite order, hence decreases Σ in the same order. This is the only place where the two parameters are used, and they are used identically, which is why the result for τ_u comes for free.

It remains to notice that the part of WL that depends on the public information at all

is $\frac{1}{2}\mathbb{E}[(\tilde{x}-x^0)'\Phi(\tilde{x}-x^0)]$, and that with the a's held fixed $\tilde{x}-x^0 = Le$ for a matrix L that does not depend on τ_θ, τ_u. Hence this part equals $\frac{1}{2}tr(L'\Phi L\Sigma)$, where $L'\Phi L$ is positive semidefinite since $\Phi = \Lambda + B$ is positive definite, Λ being positive definite and B positive semidefinite, and therefore it is non-decreasing in Σ in the positive semidefinite order. The rest of WL does not depend on τ_θ, τ_u at all.

So, holding the a's fixed, WL is decreasing in τ_θ and in τ_u, and by the envelope theorem so is WL^T, hence proven

B Additional results

This appendix collects the results that either carry over from V17 or extend the analysis in a direction that does not involve the linkages.

B.1 Non-price-contingent strategies

In this case, a strategy for firm i is a mapping from signals I_i into outputs: $X_i(I_i)$. This is the model considered in Vives (1988) and in Section 6.1 of Angeletos and Pavan (2007). The equilibrium follows immediately from the optimization problem of firm i, $\max_{x_{i1},x_{i2}} \mathbb{E}[\pi_i|I_i]$. It follows that, under the same distributional assumptions about model parameters, there is a unique Bayesian Cournot equilibrium and it is symmetric and linear. For $\rho = 0$ it can be written in the matrix form of Section 4.2 as

$$x_i = \Phi^{-1}(\alpha - \hat{\theta}) - C^C(s_i - \hat{\theta}), \qquad C^C = \tau_\epsilon(\tau_\epsilon\Phi + \tau\Lambda)^{-1},$$

which differs from the team schedules with price conditioning, $C^T = \tau_\epsilon(\tau\Phi + \tau_\epsilon\Lambda)^{-1}$, by the exchange of the roles of Φ and Λ. Indeed, with $x_i = A - Cs_i - D\hat{\theta}$ and $\mathbb{E}[\theta|I_i] = \gamma^M s_i + (1-\gamma^M)\hat{\theta}$ the F.O.C. $\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|I_i] = 0$ gives $(\Lambda + \gamma^M B)C = \gamma^M I$ for the coefficients on s_i, whose solution is C^C, since $\tau_\epsilon(\Lambda + \gamma^M B) = \gamma^M(\tau_\epsilon\Phi + \tau\Lambda)$, and $\Phi D = (1-\gamma^M)(I - BC)$

for the coefficients on $\hat{\theta}$, whose solution is $D = \Phi^{-1} - C$. For $\lambda_3 = 0$, in the representation $x_{ij} = K_\phi^{-1}[\phi_{-j}(\alpha_j - \gamma_{jj}^C s_{ij} - (1 - \gamma_{jj}^C)\hat{\theta}_j) - \beta_3(\ldots)]$, this gives the main weights

$$\gamma_{jj}^C = \frac{K_\phi(\phi_{-j}\tau_\epsilon + \lambda_{-j}\tau)\tau_\epsilon}{\phi_{-j}[(\phi_1\tau_\epsilon + \lambda_1\tau)(\phi_2\tau_\epsilon + \lambda_2\tau) - (\beta_3\tau_\epsilon)^2]},\ j \in \{1, 2\}.$$

Market equilibrium here is also team-efficient, since both solutions have associated F.O.C. of the form $E[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|I_i] = 0$, and, unlike in the market game, firms do *not* condition on prices. The details are the following.

The firm's maximization problem is the following:

$$\max_{x_{i1}, x_{i2}} \mathbb{E}[p_1 x_{i1} + p_2 x_{i2} - \theta_1 x_{i1} - \theta_2 x_{i2} - 0.5\lambda_1 x_{i1}^2 - 0.5\lambda_2 x_{i2}^2 - \lambda_3 x_{i1} x_{i2}|I_i]$$

First-order conditions for a firm in market equilibrium are indeed:

$$\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|I_i] = 0, j \in \{1, 2\}$$

In the team solution, the planner maximizes the following welfare, subject to linear decentralized strategies

$$\mathbb{E}\left[\sum_{j=1}^{2}(\alpha_j + u_j - 0.5\beta_j\tilde{x}_j)\tilde{x}_j - \beta_3\tilde{x}_1\tilde{x}_2 - \int_0^1 \left[\sum_{j=1}^{2}(\theta_j x_{ij} + 0.5\lambda_j x_{ij}^2) + \lambda_3 x_{i1}x_{i2}\right] di\right]$$

If we maximize this w.r.t. the parameters in x_{ij} associated with z_j, we simply get the conditions

$$\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|z_j, z_{-j}] = 0, j \in \{1, 2\}$$

This is because the derivatives of $x_{ij}, \tilde{x}_j$ here are the same (public information).

When it comes to private information, things become slightly trickier:

$$\mathbb{E}[p_j\theta_k - \mathrm{MC}_j(x_{i1}, x_{i2})s_{ik}] = 0, j, k \in \{1, 2\}$$

Here the derivatives of $x_{ij}, \tilde{x}_j$ are different because of the aggregation of private information.

But let us notice that

$$\mathbb{E}[p_j(s_{ik} - \theta_k)] = 0, j, k \in \{1, 2\}$$

And hence our F.O.C.s could be written as

$$\mathbb{E}[(p_j - \mathrm{MC}_j(x_{i1}, x_{i2}))s_{ik}] = 0, j, k \in \{1, 2\}$$

Or, since we are in a linear-Gaussian world, this is equivalent to

$$\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|s_{ij}, s_{i(-j)}] = 0, j \in \{1, 2\}$$

So the joint F.O.C.s of the social planner are the same as in market equilibrium, proven

$$\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|I_i] = 0, j \in \{1, 2\}$$

Repeating an exercise similar to the derivation of the team-solution form for market schedules with these F.O.C.s will give us the desired answer.

B.1.1 Proof of Proposition 1

This is just the observation above restated. The F.O.C. of the firm in the Cournot market and the F.O.C. of the team problem are both of the form $\mathbb{E}[p_j - \mathrm{MC}_j(x_{i1}, x_{i2})|I_i] = 0$, and neither of them contains prices as conditioning variables, so the two coincide. Nothing in this argument uses the values of β_3, λ_3 or ρ: they enter MC_j and p_j but enter them identically in the two problems, hence proven

B.2 Heterogeneous information precision

Here I relax the assumption about homogeneous precision for private and public signals, allowing for heterogeneity: $\mathbb{V}[\varepsilon_{i1}] = \tau_{\epsilon 1}^{-1} \neq \mathbb{V}[\varepsilon_{i2}] = \tau_{\epsilon 2}^{-1}$, $\mathbb{V}(\theta_1 - \mathbb{E}[\theta_1|z_1, z_2]) = \tau_1^{-1} \neq \mathbb{V}(\theta_2 - \mathbb{E}[\theta_2|z_1, z_2]) = \tau_2^{-1}$; however, I still require that $\theta_1 \perp \theta_2$, $\omega_1 \perp \omega_2$, so that $\rho = 0$.

In that case nothing changes qualitatively. The main intuition behind it is simple: since $\theta_1 \perp \theta_2$ and shocks are independent, there is no connection between signals about θ_1 and signals about θ_2, so firms can separate their losses into those that come from imperfect information about θ_1 and those about θ_2 and minimize separately. Let me stress that this separation does not require $\lambda_3 = 0$: the structural linkages couple the two schedules of a firm, but each of the two parts of the losses depends only on how both schedules use the information about one cost parameter (see Appendix A.3). Hence the weights on the information about θ_j are given by the formulas of Section 4.2 with τ_ϵ, τ replaced by $\tau_{\epsilon j}, \tau_j$. Denote $F_k^j = \phi_k \tau_j + \lambda_k \tau_{\epsilon j}$, $k \in \{1, 2, 3\}$, and $K_F^j = F_1^j F_2^j - (F_3^j)^2$; then the exact equations for γ^T, a^T's are the following:

$$\gamma_{jj}^T = \frac{K_\lambda F_{-j}^j \tau_{\epsilon j}}{\lambda_{-j} K_F^j}, \qquad \gamma_{(-j)j}^T = \frac{K_\lambda F_3^j \tau_{\epsilon j}}{\lambda_3 K_F^j}, \qquad a_{-j}^T = \frac{F_3^j \tau_{\epsilon j}}{K_F^j}, \qquad j \in \{1, 2\},$$

where $\gamma_{(-j)j}^T$ (defined for $\lambda_3 \neq 0$) and a_{-j}^T are the weight and the side weight on the private signal s_{ij} in the schedule $X_{i(-j)}^T$. In the matrix form of Section 4.2, the j-th column of C^T is the j-th column of $\tau_{\epsilon j}(\tau_j \Phi + \tau_{\epsilon j} \Lambda)^{-1}$. For $\lambda_3 = 0$ we have $F_3^j = \beta_3 \tau_j$, and for $\beta_3 = 0$ we have $F_3^j = \lambda_3(\tau_j + \tau_{\epsilon j})$.

In the representation of the welfare losses of Subsection A.3 with heterogeneous precisions we see that this function is separable in minimization into two sums, where the first one contains only c_1 and $\tau_{\epsilon 1}, \tau_1$, while the second one contains only c_2 and $\tau_{\epsilon 2}, \tau_2$, and this is true for any β_3, λ_3. Each of them is exactly the corresponding part of the losses with homogeneous precisions, hence if we take the result of heterogeneous precision and remove the indices we should get the answers from Section 4.2, and, the other way round, the weights on the

information about θ_k are the ones of Section 4.2 with τ_ϵ, τ replaced by $\tau_{\epsilon k}, \tau_k$, which gives the formulas above. Hence we would not have any qualitative difference.

The only new feature is that the side weights of the two schedules are no longer equal, since each of them is determined by the precision of the information that it brings from the other market. All statements of Sections 4 and 5 about the weights therefore carry over group by group: the weights on the information about θ_j behave in $\tau_{\epsilon j}, \tau_j$ and in the linkages exactly as in the homogeneous case and do not depend on $\tau_{\epsilon(-j)}, \tau_{-j}$ at all; in particular, Proposition 2 holds with $\gamma_j^M = \frac{\tau_{\epsilon j}}{\tau_{\epsilon j}+\tau_j}$. The same is true for the team welfare loss, which is the sum of two parts, each of them decreasing in its own precisions.

The same separation holds for the market, whose loss is

$$\mathrm{WL}^M = \frac{1}{2}\sum_{j=1}^{2} \frac{(\Lambda^{-1})_{jj}\tau_{\epsilon j} + (\Phi^{-1})_{jj}\tau_j}{(\tau_j + \tau_{\epsilon j})^2},$$

and the argument of the proof of Proposition 5 shows that more precise private information about θ_j increases WL^M if and only if $\frac{\tau_{\epsilon j}}{\tau_j} < 1 - \frac{2}{R_j}$, where $R_j = \frac{(\Lambda^{-1})_{jj}}{(\Phi^{-1})_{jj}} = \frac{\lambda_{-j}K_\phi}{\phi_{-j}K_\lambda}$. Here R_j is decreasing in $|\beta_3|$ if $\lambda_3 = 0$, increasing in $|\lambda_3|$ if $\beta_3 = 0$, and equal to $1 + k$ under proportionality, so Proposition 5 holds market by market.

B.3 Precision of information

Let me first notice that WL and hence γ^T's are independent of τ_u, so the only information parameters that matter here are τ_ϵ and τ. The following proposition states the most important properties of $\gamma^M, \gamma^T, \mathrm{WL}^M, \mathrm{WL}^T$ with respect to them, where the latter are the expected losses in market equilibrium and the team solution; Proposition 5 describes exactly when WL^M is increasing in τ_ϵ, and Proposition 8 gives the counterpart for correlated public signals.

Proposition 10. *Let $0 < \tau_\epsilon < \infty$ and $\rho = 0$. Then for any β_3, λ_3:*

(i) γ_j^M's and γ_{jj}^T's are increasing in τ_ϵ and decreasing in τ, depending only on $\frac{\tau_\epsilon}{\tau}$;

(ii) $\mathrm{WL}^M, \mathrm{WL}^T$ *are decreasing in* τ*, while* WL^T *is decreasing in* τ_ϵ*, but* WL^M *can be increasing or decreasing in* τ_ϵ*, depending on the model's parameters.*

All these facts, except the exact formula for γ^T, were also observed in V17 for a single market, so here I have just generalized his findings to a more general production and consumption structure, and there is nothing new in the intuition behind them. One important thing that was very briefly mentioned in the original paper is that the reason for WL^M to be possibly increasing in τ_ϵ is that "an increase in the precision of private information will increase the response of an agent to his private signal, and this indirect effect may dominate". This mostly happens when $\frac{\tau_\epsilon}{\tau}$ is low enough, and then suboptimal overreaction to private information can cause more expected potential losses from distributive inefficiency than potential gains from a decrease in aggregate inefficiency. Public information, in contrast, is never harmful: more precise public information decreases the welfare losses in both solutions.

B.3.1 Proof of Proposition 10

i)For γ_j^M's the result is obvious. For γ_{jj}^T's let us notice that F_1, F_2, F_3 are homogeneous of degree one in (τ, τ_ϵ), hence γ_{jj}^T are homogeneous of degree zero and depend only on $r = \frac{\tau_\epsilon}{\tau}$. Moreover, direct computation gives

$$(\gamma_{11}^T)^{-1} = 1 + \frac{v'\Phi v}{K_\lambda(\phi_2 + \lambda_2 r)} + \frac{\lambda_2 K_\phi}{K_\lambda r(\phi_2 + \lambda_2 r)}, \qquad v = (\lambda_2, -\lambda_3)',$$

where $v'\Phi v, K_\phi > 0$, since Φ is positive definite. Both fractions are obviously decreasing in r, hence γ_{11}^T is increasing in r, i.e. increasing in τ_ϵ and decreasing in τ, and the same is true for γ_{22}^T based on the symmetry of the model.

ii)As for WL^T, I use the representation of the welfare losses of Subsection A.3 with

$\tau_{\epsilon 1} = \tau_{\epsilon 2} = \tau_\epsilon$, $\tau_1 = \tau_2 = \tau$:

$$\mathrm{WL}(C; \tau_\epsilon, \tau) = \frac{1}{2} \sum_{k=1}^{2} \left(\frac{c_k' \Lambda c_k}{\tau_\epsilon} + \frac{d_k' \Lambda \Phi^{-1} \Lambda d_k}{\tau} \right).$$

For fixed coefficients C both quadratic forms are nonnegative, hence $\mathrm{WL}(C; \tau_\epsilon, \tau)$ is decreasing in τ_ϵ and τ. Moreover, at the team solution C^T they are strictly positive, since the k-th entries of c_k and d_k are $\frac{\lambda_{-k}\gamma_{kk}^T}{K_\lambda} > 0$ and $\frac{\lambda_{-k}(1-\gamma_{kk}^T)}{K_\lambda} > 0$ by Proposition 2. Then for $\tau_\epsilon' > \tau_\epsilon$ we have

$$\mathrm{WL}^T(\tau_\epsilon') \leq \mathrm{WL}(C^T(\tau_\epsilon); \tau_\epsilon', \tau) < \mathrm{WL}(C^T(\tau_\epsilon); \tau_\epsilon, \tau) = \mathrm{WL}^T(\tau_\epsilon),$$

and the same for τ, so WL^T is decreasing in τ_ϵ, τ.

As for WL^M, by Subsection A.3

$$\mathrm{WL}^M = \frac{\mathrm{tr}(\Lambda^{-1})\tau_\epsilon + \mathrm{tr}(\Phi^{-1})\tau}{2(\tau + \tau_\epsilon)^2} = \frac{a\tau_\epsilon}{(\tau + \tau_\epsilon)^2} + \frac{b}{\tau + \tau_\epsilon},$$

where $a = \frac{1}{2}\mathrm{tr}(\Lambda^{-1}) - \frac{1}{2}\mathrm{tr}(\Phi^{-1}) > 0$, since $\Phi - \Lambda = B$ is positive semidefinite and nonzero, and $b = \frac{1}{2}\mathrm{tr}(\Phi^{-1}) > 0$. Hence it is definitely decreasing in τ, while I can't definitely say that for τ_ϵ, since its derivative is $\frac{(a-b)\tau - (a+b)\tau_\epsilon}{(\tau+\tau_\epsilon)^3}$.

And indeed, notice that if τ_ϵ is small enough and $a > b$, the function is definitely increasing, while for $\tau_\epsilon > \frac{a-b}{a+b}\tau$ it is decreasing, and $\frac{a-b}{a+b} = 1 - \frac{2\mathrm{tr}(\Phi^{-1})}{\mathrm{tr}(\Lambda^{-1})}$. The condition $a > b$ is possible for example when $\beta_3 = \lambda_3 = 0$ and $\beta_1 > \lambda_1, \beta_2 > \lambda_2$, since then $\phi_j > 2\lambda_j$ and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} > \frac{2}{\phi_1} + \frac{2}{\phi_2}$, and by continuity also for small $|\beta_3|, |\lambda_3|$. For $\lambda_3 = 0$ the formula for WL^M is exactly the one of Subsection A.1, since then $\mathrm{tr}(\Lambda^{-1}) = \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}$, while $\mathrm{tr}(\Phi^{-1}) = \frac{\phi_1+\phi_2}{K_\phi}$ always.

C Numerical simulation results

This section contains the graphs of the model variables w.r.t. the exogenous parameters for the endogenous case of public information. Some of them demonstrate so-called "completely

non-monotone" behavior, where there is no monotonicity and not even (quasi-)concavity.

Here the initial parameters were $\lambda_1 = \lambda_2 = \beta_1 = \beta_2 = \tau_\theta = \tau_u = \tau_\epsilon = 1, \beta_3 = 0.5, \lambda_3 = 0$, if not specified otherwise. $\lambda_3 = 0$ always, unless it is the variable of interest, and for graphs where λ_3 varies it is instead assumed that $\beta_3 = 0$. The precisions are plotted on a logarithmic scale, and the linkages over the whole range $0 \leq \beta_3 < \sqrt{\beta_1\beta_2}$ (resp. $0 \leq \lambda_3 < \sqrt{\lambda_1\lambda_2}$) on which B (resp. Λ) is positive definite. Each curve is computed on a grid of 160 to 200 points. For the market equilibrium, the system of Subsection A.5 was solved at each point from many starting points, and only one solution with $a_{11}, a_{22} > 0$ and $a_{11}a_{22} > a_{12}a_{21}$ was found; for the team solution, the loss of Subsection A.6 was minimized at each point from many starting points, and the global minimizer changes continuously along every curve.

C.1 Examples establishing Facts 1 and 2

Each part of Facts 1 and 2 is an existence statement, and Table 2 lists, for each of them, the figure of Subsection C.2 that establishes it; the parameter values are given with the figure, the default values being the ones stated above.

The figures show the variables only on the half-line $\beta_3 \geq 0$ (resp. $\lambda_3 \geq 0$). Since by the joint parity of Section 7.3 main weights are even and side weights are odd in the linkage when the other linkage is zero, a variable that rises and then falls, or falls and then rises, on the half-line is neither quasi-concave nor quasi-convex on the whole domain, and all the figures referred to for part (i) of both Facts are of this kind.

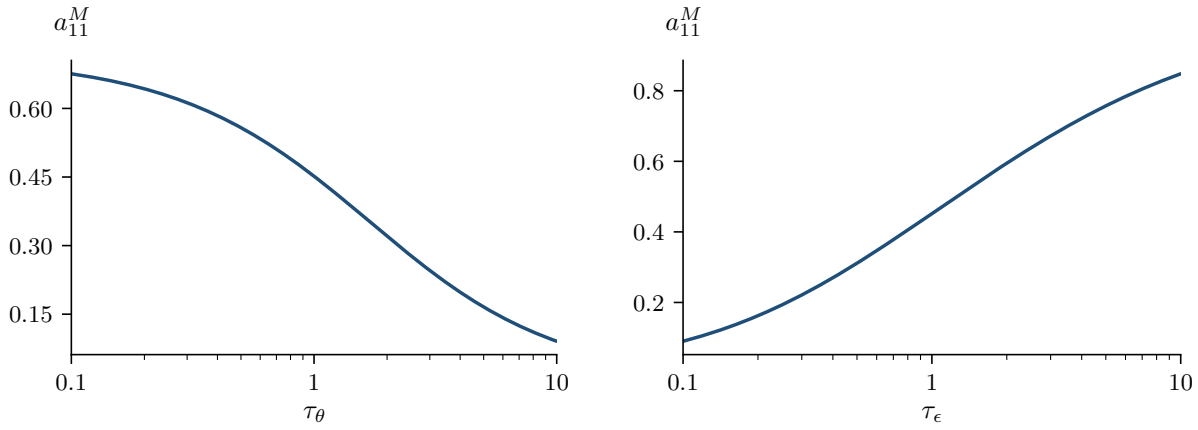
Table 2: Examples establishing Facts 1 and 2

Statement	Figure
Fact 1(i), β_3	Figure 4(a)
Fact 1(i), λ_3	Figure 4(b)
Fact 1(ii)	Figure 2(b)
Fact 1(iii)	Figure 3
Fact 1(iv)	Figure 6(a)
Fact 2(i), β_3	Figures 10(a), 11(a)
Fact 2(i), λ_3	Figures 10(b), 11(b)
Fact 2(ii), τ_θ	Figure 7(a)
Fact 2(ii), τ_ϵ	Figure 7(b)
Fact 2(iii), τ_θ	Figure 8(a)
Fact 2(iii), τ_ϵ	Figure 8(b)

C.2 Figures

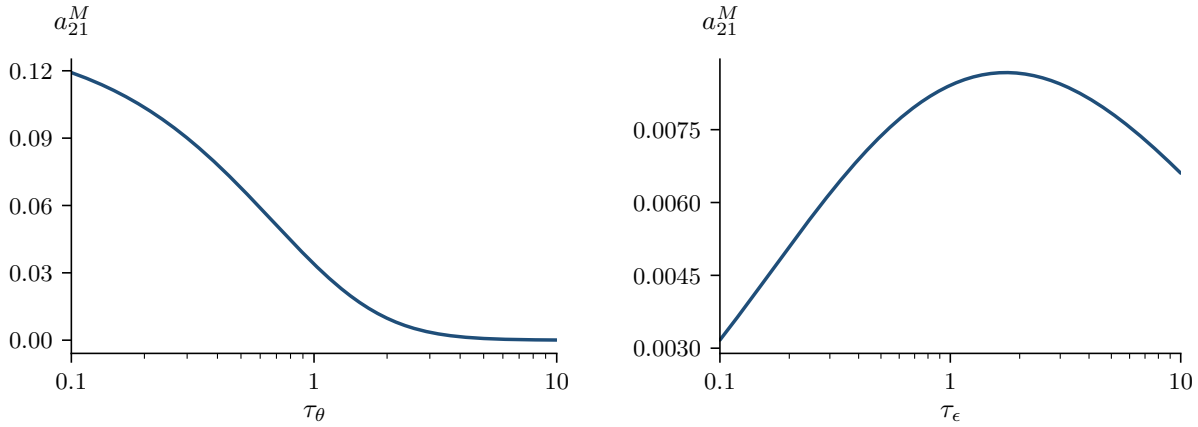
Besides the examples of Table 2, the figures illustrate the Numerical Propositions of Section 7.4: Figures 1 and 2(a) for Numerical Proposition 1, Figure 5 for Numerical Proposition 2, and Figures 6(b) and 12 for Numerical Proposition 3, while Figure 9 illustrates Proposition 9.

Figure 1: a_{11}^M w.r.t. information parameters



(a) a_{11}^M w.r.t. τ_θ (b) a_{11}^M w.r.t. τ_ϵ

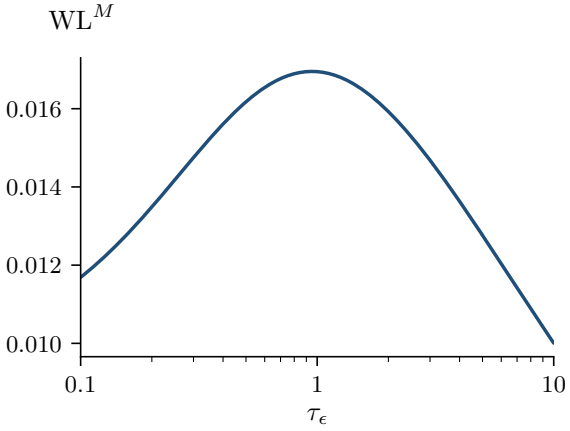
Figure 2: a_{21}^M w.r.t. information parameters



(a) a_{21}^M w.r.t. τ_θ (b) a_{21}^M w.r.t. τ_ϵ

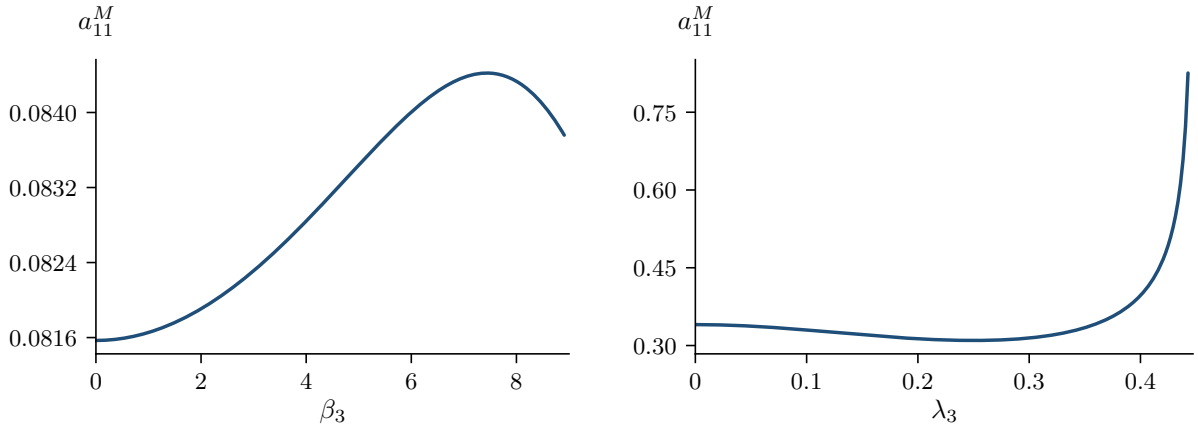
For subfigure (b) the parameters are: $\lambda_1 = 5, \lambda_2 = 7, \beta_1 = 6, \beta_2 = 1, \beta_3 = 1, \tau_\theta = 0.2, \tau_u = 10$

Figure 3: WL^M w.r.t. τ_ϵ



The parameters are: $\lambda_1 = 2, \lambda_2 = 1, \beta_1 = \beta_2 = 20, \beta_3 = 0, \tau_\theta = 5, \tau_u = 0.5$

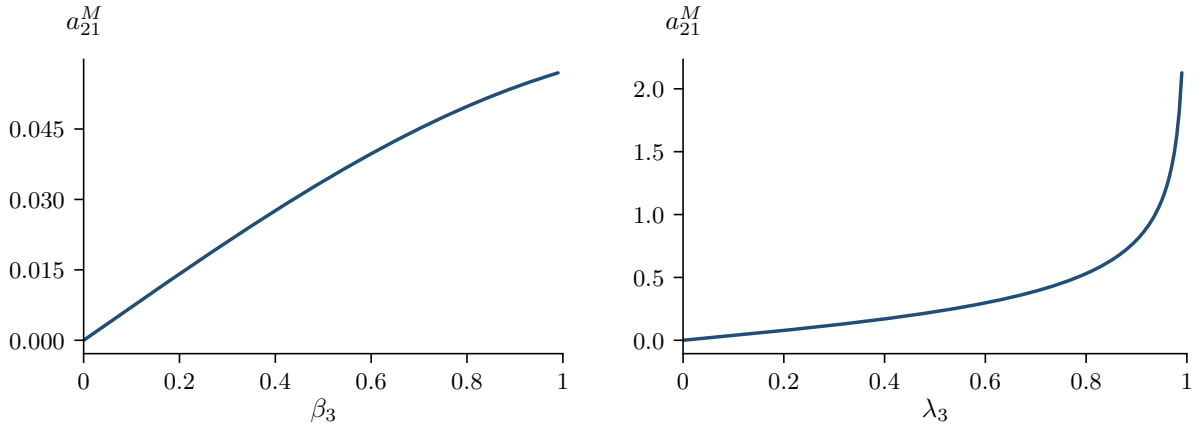
Figure 4: a_{11}^M w.r.t. consumption/production parameters



(a) a_{11}^M w.r.t. β_3 (b) a_{11}^M w.r.t. λ_3

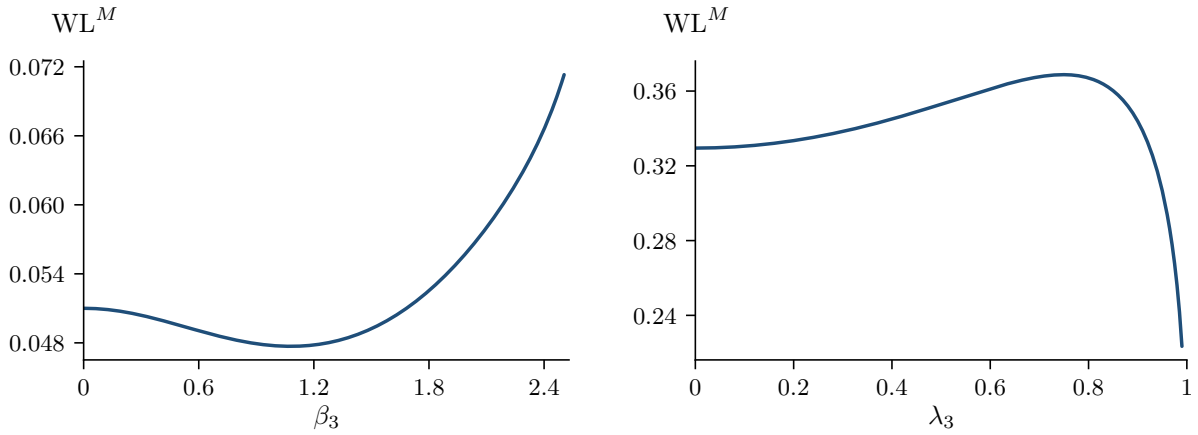
For subfigure (a) the parameters are: $\lambda_1 = 4, \lambda_2 = 5, \beta_1 = 9, \beta_2 = 9, \tau_\theta = 7, \tau_u = 10, \tau_\epsilon = 6$
For subfigure (b) the parameters are: $\lambda_1 = 1, \lambda_2 = 0.2, \beta_1 = 5, \beta_2 = 0.2, \tau_\theta = 1, \tau_u = 3, \tau_\epsilon = 5$

Figure 5: a_{21}^M w.r.t. consumption/production parameters



(a) a_{21}^M w.r.t. β_3 (b) a_{21}^M w.r.t. λ_3

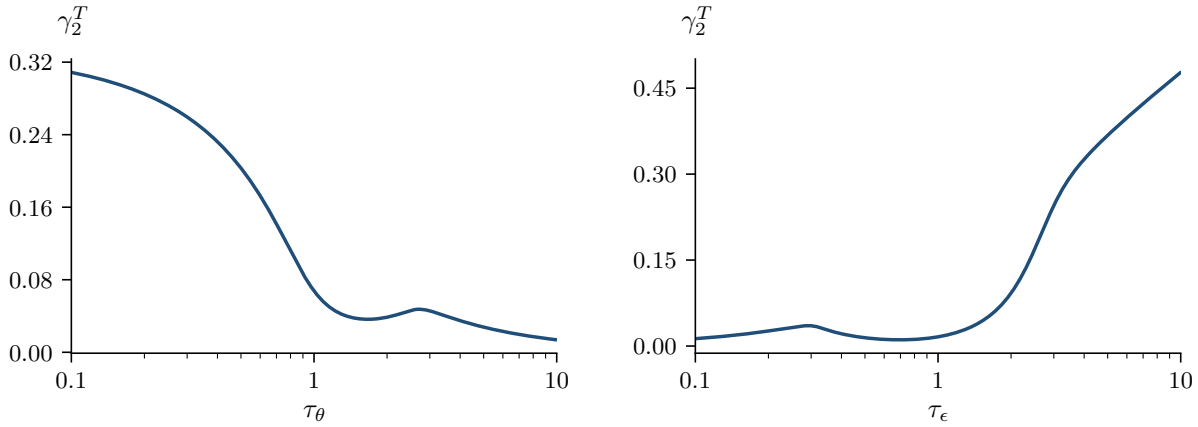
Figure 6: WL^M w.r.t. consumption/production parameters



(a) WL^M w.r.t. β_3 (b) WL^M w.r.t. λ_3

For subfigure (a) the parameters are:
$\lambda_1 = 0.1, \lambda_2 = 2.2, \beta_1 = 0.8, \beta_2 = 8, \tau_\theta = 2.9, \tau_u = 11, \tau_\epsilon = 5$

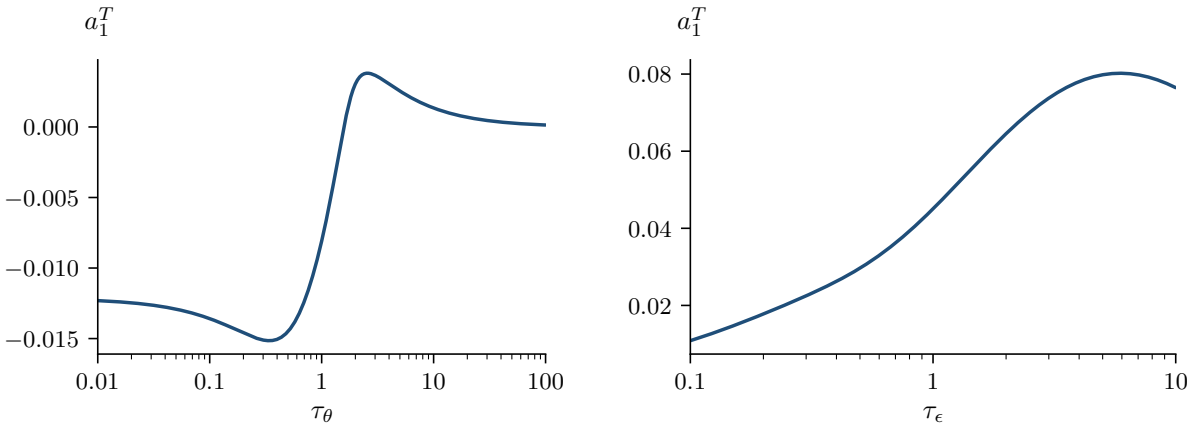
Figure 7: γ_2^T w.r.t. information parameters



(a) γ_2^T w.r.t. τ_θ (b) γ_2^T w.r.t. τ_ϵ

For subfigure (a) the parameters are:
$\lambda_1 = 1.083, \lambda_2 = 1.819, \beta_1 = 5.677, \beta_2 = 3.607, \beta_3 = 1.814, \tau_u = 2.352, \tau_\epsilon = 0.375$
For subfigure (b) the parameters are:
$\lambda_1 = 1.861, \lambda_2 = 2.12, \beta_1 = 5.483, \beta_2 = 4.809, \beta_3 = -3.022, \tau_\theta = 3.009, \tau_u = 4.06$

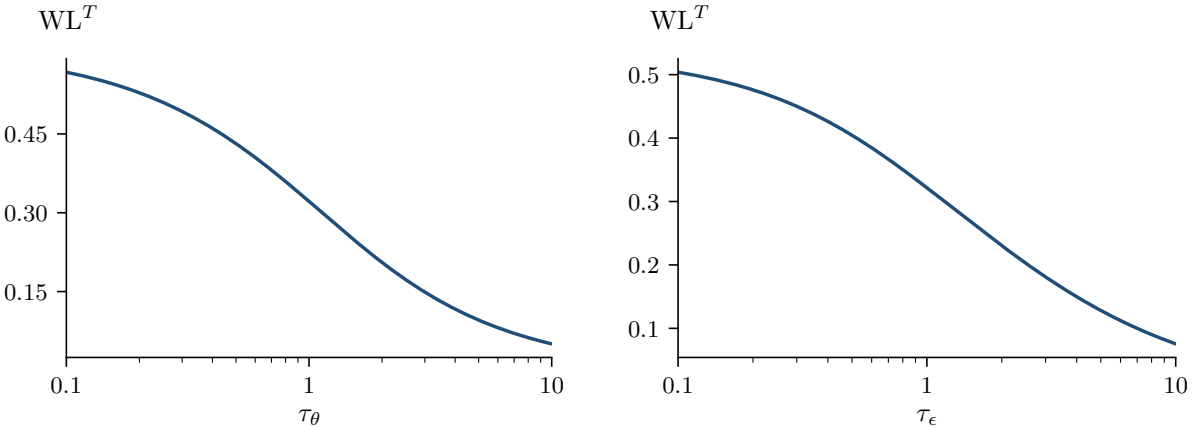
Figure 8: a_1^T w.r.t. information parameters



(a) a_1^T w.r.t. τ_θ (b) a_1^T w.r.t. τ_ϵ

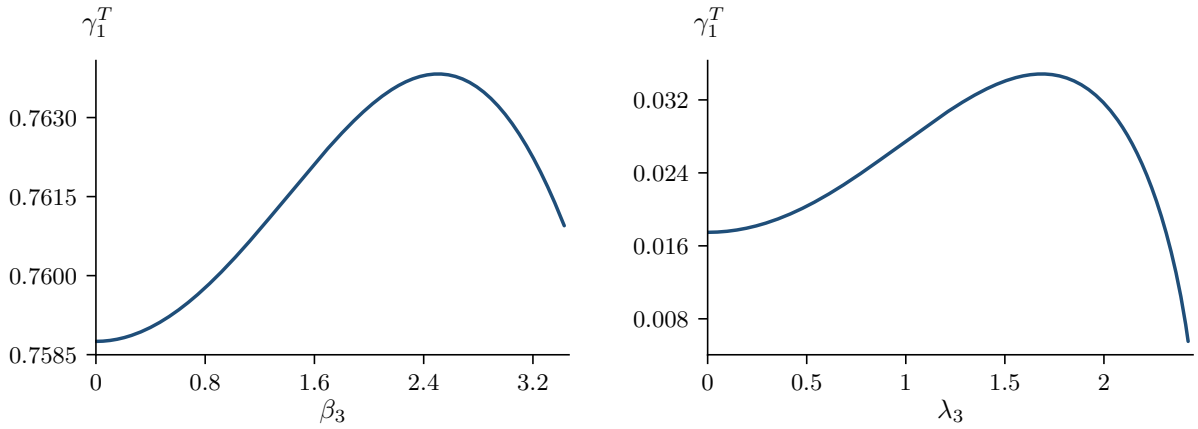
For subfigure (a) the parameters are:
$\lambda_1 = 3.5, \lambda_2 = 0.5, \beta_1 = 4, \beta_2 = 3, \beta_3 = 1, \tau_u = 1, \tau_\epsilon = 0.35$

Figure 9: WL^T w.r.t. information parameters



(a) WL^T w.r.t. τ_θ (b) WL^T w.r.t. τ_ϵ

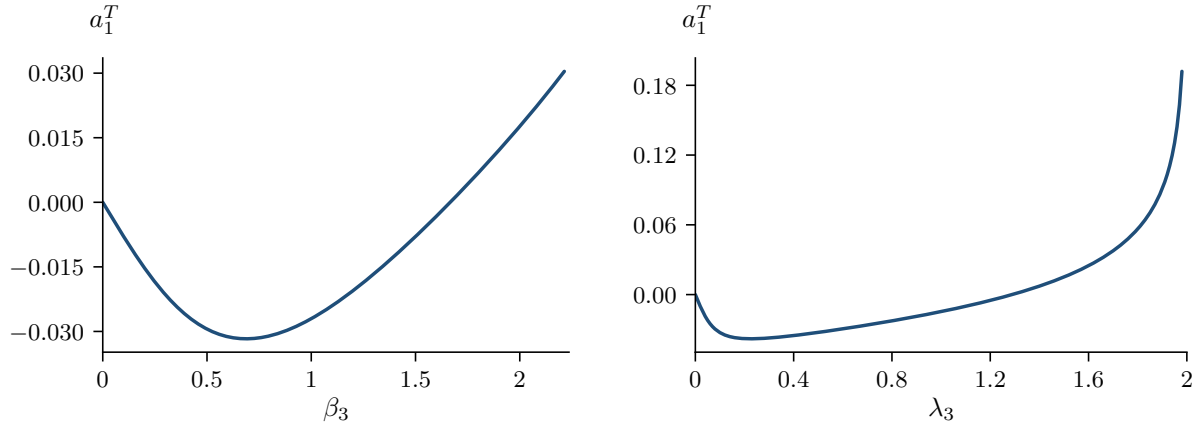
Figure 10: γ_1^T w.r.t. consumption/production parameters



(a) γ_1^T w.r.t. β_3 (b) γ_1^T w.r.t. λ_3

For subfigure (a) the parameters are: $\lambda_1 = 7, \lambda_2 = 6, \beta_1 = 4, \beta_2 = 3, \tau_\theta = 1, \tau_u = 6, \tau_\epsilon = 9$
For subfigure (b) the parameters are:
$\lambda_1 = 3, \lambda_2 = 2, \beta_1 = 5, \beta_2 = 10, \tau_\theta = 5, \tau_u = 20, \tau_\epsilon = 0.2$

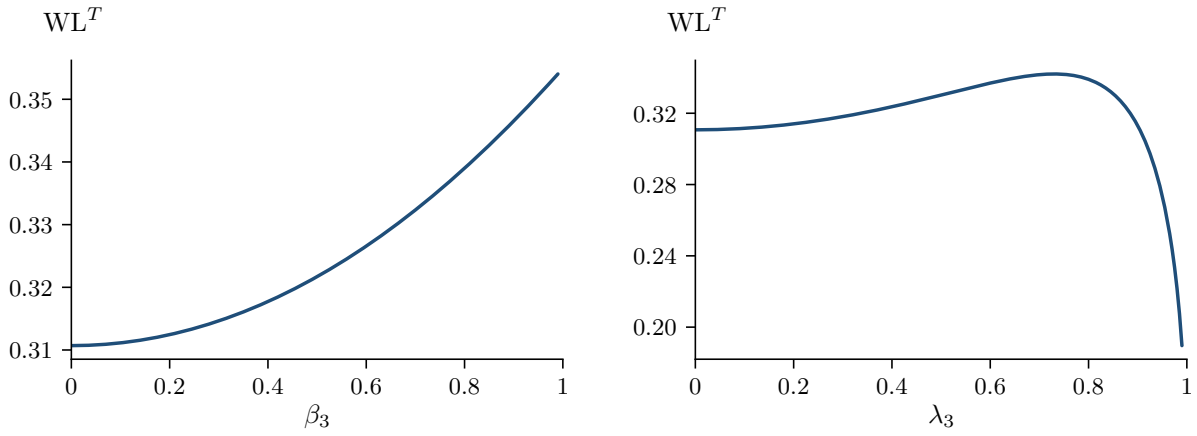
Figure 11: a_1^T w.r.t. consumption/production parameters



(a) a_1^T w.r.t. β_3 (b) a_1^T w.r.t. λ_3

For subfigure (a) the parameters are:
$\lambda_1 = 5, \lambda_2 = 0.2, \beta_1 = 5, \beta_2 = 1, \tau_\theta = 0.5, \tau_u = 1, \tau_\epsilon = 10$
For subfigure (b) the parameters are: $\lambda_1 = 2, \lambda_2 = 2, \beta_1 = 1, \beta_2 = 6, \tau_\theta = 4, \tau_u = 9, \tau_\epsilon = 4$

Figure 12: WL^T w.r.t. consumption/production parameters



(a) WL^T w.r.t. β_3

(b) WL^T w.r.t. λ_3