Online Appendix to "Competitive Persuasion with a Noisy Judge"

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Main paper: karpov-igor.github.io/Karpov_JMP.pdf

Overview

This Online Appendix collects the results that the paper states only in passing or not at all. Numbers with the prefix OA refer to this document; all other numbers refer to the paper. Section OA.1 adds to Section 3: the limit of pooling, the regions and payoffs under unequal priors, and how informativeness and selection reversals respond to the prior. Section OA.2 adds to Section 4: costly signals, under the variance cost and for the robust implementation of full revelation, and the remaining case of the triangular rules. Section OA.3 adds to Section 5: refinements of Lemma 11 and Proposition 9, the best fully revealing rules, the random pass mark, continuous rules, the noiseless equilibrium, and a weaker form of condition (ii) of Theorem 3. Section OA.4 contains the extensions: reverse S-shaped rules, the boundary of the well S-shaped class, and a single agent. Each proof follows its statement.

OA.1. Equilibrium under Well S-Shaped Rules

OA.1.1 The limit of pooling

Lemma OA.1. *Let $p \in (0,1)$, let $v_n(x)$ be well S-shaped for all $n \in \mathbb{N}$, and let $(\{0, a_n\}; \{0, a_n\})$ be the equilibrium for v_n and priors (p, p). If $\mathcal{E}_n(x) \to \infty$ uniformly on $(0,1)$, then $a_n \to p$.*

Proof. Since $\mathcal{E}_n(x) \to \infty$ uniformly on $(0,1)$ and $\mathcal{E}_n(x)$ is continuous on $(0,1]$, $\varepsilon_n := \sup_{x \in (0,1]} \frac{1}{\mathcal{E}_n(x)} \to 0$. Hence $p < \overline{p}_n = 1 - \frac{1}{\mathcal{E}_n(1)}$ for large n, and then, by Theorem 1, $a_n = \frac{p}{1 - 1/\mathcal{E}_n(a_n)} \in \left[p, \frac{p}{1-\varepsilon_n}\right]$, so $a_n \to p$. □

OA.1.2 Unequal priors: regions and further results

The regions of Proposition 2 are bounded by best replies to the three forms of signals, listed below with their values in Example 3.

	definition	*Example 3*
$\overline{p}$	$1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)}$: full revelation for $p_1, p_2 \geq \overline{p}$	$\frac{2}{3}$
$b(p_1)$	agent 2's best reply to $\{p_1\}$ is $\{0, b(p_1)\}$ if $p_2 < b(p_1)$, else $\{p_2\}$	$\min(\frac{3p_1}{2}, 1)$
p^*	smallest p_1 with $b(p_1) = 1$	$\frac{2}{3}$
$a(p_2)$	agent 1's best reply to $\{0, 1\}$ is $\{0, a(p_2)\}$ if $p_1 < a(p_2)$, else $\{p_1\}$	$\min(\frac{3p_2}{2}, 1)$
$\check{p}_2(p_1)$	largest p_2 for which $\{p_1\}$ is agent 1's best reply to $\{0, b(p_1)\}$	$\frac{2p_1}{3}$
$\hat{p}_1(p_2)$	smallest p_1 for which $\{0, 1\}$ is agent 2's best reply to $\{0, a(p_2)\}$	$\frac{2}{3}$

With $p_2 \leq p_1$ and $p_2 < \overline{p}$, the equilibrium is

(i) $(\{p_1\}; \{0, 1\})$ if $p_1 \geq \max\{p^*, a(p_2)\}$;

(ii) $(\{p_1\}; \{0, b(p_1)\})$ if $p_1 < p^*$ and $p_2 \leq \check{p}_2(p_1)$;

(iii) $(\{0, a(p_2)\}; \{0, 1\})$ if $\hat{p}_1(p_2) \leq p_1 < a(p_2)$;

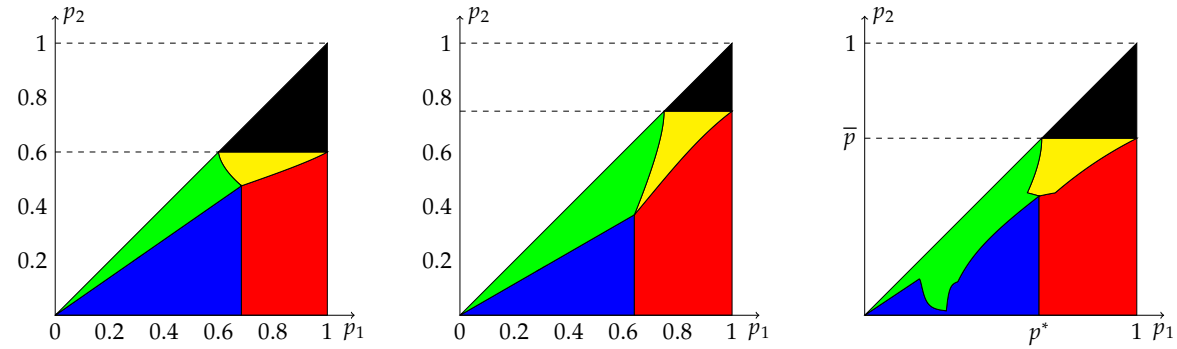
(iv) $(\{0, a\}; \{0, b\})$ with $p_1 < a \leq b < 1$ otherwise.

Figure OA.1 depicts these regions for three other rules. In Example 3, $\overline{p} = p^*$ (Figure 3): it is the knife-edge case between $\overline{p} < p^*$ and $\overline{p} > p^*$ in panels (a) and (b). The boundary $a(\cdot)$ is monotone, but $\hat{p}_1(\cdot)$ and $\check{p}_2(\cdot)$ need not be: in panel (c), neither is monotone.

In region (i), against full revelation, agent 1's value function $(1-p_2)v(s_1)+p_2v(s_1-1)$ is close to $v(s_1)$ for low p_2, so it is concave except near 0, and pooling at p_1 is optimal. Against pooling, agent 2's value function $v(s_2 - p_1)$ is convex on $[0, p_1]$ and concave on $[p_1, 1]$, and for high p_1, its concavification is supported on $\{0, 1\}$ despite the concave part. For a fixed $v(x)$, not every pair of forms is played: full revelation is a best reply to $\{0, c\}$ only if $c \geq p^*$, so $(\{0, c\}; \{0, 1\})$ is an equilibrium only if $c \geq p^*$.

Remark OA.1. Let $v(x) = v^*(x)$, for which $v'(x)$ is linear on $[0, 1]$, and $(p_1, p_2) = (0.4, 0.3)$. Then agent 2 plays $\{0, 0.6\}$ in every equilibrium, and agent 1's value function against it is linear on $[0, 0.6]$. So agent 1's equilibrium signals are exactly those with support in $[0, 0.6]$ and $\mathbb{E}[s_1^2] = 0.18$, which keep $\{0, 0.6\}$ a best reply of agent 2, for example $\{0, 0.45\}$, $\{0.3, 0.6\}$ and their mixtures.

Figure OA.1: Equilibria for unequal priors, with $\tilde{v}(x)$ given on $[0,1]$



(a) $\tilde{v}(x) = -x^{2.5}$:
$\overline{p} = 0.6 < p^* \approx 0.68$

(b) $\tilde{v}(x) = -x^4$:
$\overline{p} = 0.75 > p^* \approx 0.64$

(c) $\tilde{v}(x) = -\frac{x^3+60\max(x-0.3,0)^2}{30.4}$:
$p^* \approx 0.641 < \overline{p} \approx 0.651$

Colors are for the corresponding equilibrium type: black – $(\{0,1\};\{0,1\})$; *red* – $(\{p_1\};\{0,1\})$; *green* – $(\{0,a\};\{0,b\})$; *blue* – $(\{p_1\};\{0,b\})$; *yellow* – $(\{0,a\};\{0,1\})$.

Example OA.1. Let $v(x)$ be well S-shaped with $\tilde{v}(x) = -x^{2.5}$ on $[0,1]$, and let $p_j = 0.6$. Then agent i's equilibrium signal is non-monotone in p_i.

For the firm, the monotonicity of Proposition 4 extends to unequal priors everywhere except in region (iv) of Proposition 2.

Proposition OA.1. *Let $v(x)$ be well S-shaped with $v'(x)$ not linear near 0, and let $p_2 \le p_1$. In the interior of the full revelation region and of each of the regions (i)–(iii) of Proposition 2, both partial derivatives of u_F are positive.*

Proof. Since $v(-x) = 1 - v(x)$, $s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1) = \frac{s_1+s_2}{2} + \Psi(s_1 - s_2)$, where $\Psi(x) := x\hat{v}(x)$ is even, so the firm's expected utility is

$$u_F = \frac{p_1 + p_2}{2} + \mathbb{E}[\Psi(s_1 - s_2)].$$

For $x \in (0,1]$, $0 < \Psi'(x) = \hat{v}(x) + xv'(x) \le 2\hat{v}(x) \le 2x$, since $v'(x)$ is decreasing on $[0,1]$ and $v'(0) \le 1$, as $\frac{1}{2} \ge \hat{v}(1) \ge \frac{v'(0)}{2}$ by the concavity of $v'(x)$. Also, $\Psi(x) \le \frac{x}{2}$.

Full revelation. $u_F = \frac{p_1+p_2}{2} + \Psi(1)(p_1 + p_2 - 2p_1p_2)$, so $\partial_{p_1} u_F = \frac{1}{2} + \Psi(1)(1 - 2p_2) > \frac{1}{2} - \Psi(1) \ge 0$, and similarly for p_2.

Region (i). $u_F = \frac{p_1+p_2}{2} + (1-p_2)\Psi(p_1) + p_2\Psi(1-p_1)$, so $\partial_{p_2} u_F = \frac{1}{2} - \Psi(p_1) + \Psi(1-p_1) > 0$ and, as $p_2 \le p_1$ and $\Psi'(x) \le 2x$,

$$\partial_{p_1} u_F = \frac{1}{2} + (1 - p_2)\Psi'(p_1) - p_2\Psi'(1 - p_1) > \frac{1}{2} - 2p_1(1 - p_1) \ge 0.$$

Region (ii). Let $b = b(p_1)$ and $y := b - p_1$. By Step 1 of the proof of Proposition 2, $0 < y < p_1$ and $y' = -\frac{\partial_{p_1}\varphi}{\varphi'} > 0$. Then

$$u_F = \frac{p_1 + p_2}{2} + \Psi(p_1) - \frac{p_2}{b}\big(\Psi(p_1) - \Psi(y)\big),$$

where $0 < \Psi(p_1) - \Psi(y) \leq \frac{p_1}{2} < \frac{b}{2}$. Hence $\partial_{p_2} u_F > 0$ and

$$\partial_{p_1} u_F = \frac{1}{2} + \Big(1 - \frac{p_2}{b}\Big)\Psi'(p_1) + \frac{p_2}{b}\Psi'(y)y' + \frac{p_2(1+y')}{b^2}\big(\Psi(p_1) - \Psi(y)\big) > 0.$$

Region (iii). Let $a = a(p_2)$, which is non-decreasing by Step 2 of the proof of Proposition 2, with $p_2 \leq p_1 < a$ and $a \geq p^* > \frac{1}{2}$. Then

$$u_F = \frac{p_1 + p_2}{2} + p_2\Psi(1) + \frac{p_1}{a}K, \quad \text{where} \quad K := (1 - p_2)\Psi(a) + p_2\big(\Psi(1-a) - \Psi(1)\big),$$

so $\partial_{p_1} u_F = \frac{1}{2} + \frac{K}{a} \geq \frac{1}{2} - \frac{p_2\Psi(1)}{a} > 0$ and, with $K_a := \partial_a K$,

$$\partial_{p_2} u_F = \frac{1}{2} + \Psi(1) + \frac{p_1}{a}\big(\Psi(1-a) - \Psi(1) - \Psi(a)\big) + \frac{p_1 a'}{a^2}(aK_a - K).$$

The first three terms sum to at least $\frac{1}{2} - \Psi(a) > 0$, and

$$aK_a - K = (1 - p_2)a^2 v'(a) + p_2\big(\hat{v}(1) - \hat{v}(1-a) - a(1-a)v'(1-a)\big) \geq 0,$$

since, by the concavity of $v'(x)$ and $a \geq \frac{1}{2}$,

$$\hat{v}(1) - \hat{v}(1-a) = \int_{1-a}^{1} v'(x)dx \geq \frac{a}{2}v'(1-a) \geq a(1-a)v'(1-a).$$

□

Example OA.2. Let $v(x) = \frac{1}{2} + (20x - x^{20})/38$ on $[0,1]$, so that $\tilde{v}(x) = -x^{20}$ and $v(1) = 1$, and let $p_1 = 0.924$. Then u_F strictly decreases in p_2 on $[0.839, 0.873]$, where the equilibrium is in region (iv): once agent 2 stops revealing fully, his high signal b decreases in p_2.

Computation for Example OA.2. Both signals solve the indifference conditions $N = 0$ from (N), which I solved numerically. For instance, for $p_2 = 0.84, 0.86, 0.873$, $(a, b) \approx (0.9375, 0.9996)$, $(0.9406, 0.9925)$ and $(0.9435, 0.9876)$, and $u_F \approx 0.96145, 0.96092$ and

0.96076. □

For the agents, monotonicity in the priors is trivial under equal priors, as each is selected with probability $\frac{1}{2}$. Under unequal priors, since the game is constant-sum, each agent's equilibrium payoff $u_i(p_1, p_2) = \mathbb{E}_{G_1,G_2}[v(s_i - s_j)]$ is well defined for any selection rule, whether or not the equilibrium is unique.

Lemma OA.2. *For any selection rule, $u_i(p_1, p_2)$ is non-decreasing in p_i and non-increasing in p_j. If $v(x)$ is strictly increasing, in particular if it is well S-shaped, $u_i(p_1, p_2)$ is strictly increasing in p_i and strictly decreasing in p_j.*

Proof. Write $u_i(p_i, p_j)$ for agent i's equilibrium payoff when his prior is p_i and agent j's prior is p_j. For $G_j \in \mathcal{G}(p_j)$, let $\overline{v}_{G_j}(x)$ be agent i's value function against G_j and $\overline{v}^{cv}_{G_j}$ its concavification on $[0, 1]$. By Kamenica and Gentzkow (2011, Corollary 2), $\overline{v}^{cv}_{G_j}(p_i) = \max_{G' \in \mathcal{G}(p_i)} \mathbb{E}_{G'}[\overline{v}_{G_j}]$ is agent i's best payoff against G_j, where the maximum is attained as $\overline{v}_{G_j}$ is continuous. The game is constant-sum, so agent j's equilibrium signal G^*_j minimizes i's payoff:

$$u_i(p_i, p_j) = \min_{G_j \in \mathcal{G}(p_j)} \overline{v}^{cv}_{G_j}(p_i) = \overline{v}^{cv}_{G^*_j}(p_i).$$

Step 1: each $\overline{v}^{cv}_{G_j}$ is non-decreasing on $[0, 1]$, and strictly increasing if $v(x)$ is. As $v(x)$ is increasing, so is $\overline{v}_{G_j}$, so the constant $\overline{v}_{G_j}(1)$ is a concave majorant of $\overline{v}_{G_j}$, and $\overline{v}^{cv}_{G_j} \leq \overline{v}_{G_j}(1) = \overline{v}^{cv}_{G_j}(1)$, as $\{1\}$ is the only signal with mean 1. If $v(x)$ is strictly increasing, so is $\overline{v}_{G_j}$, and every $G' \in \mathcal{G}(x)$ with $x < 1$ puts positive mass on $[0, 1)$, so $\overline{v}^{cv}_{G_j}(x) < \overline{v}^{cv}_{G_j}(1)$. For $0 \leq x < y \leq 1$, write $y = \lambda x + (1 - \lambda)$ with $\lambda = \frac{1-y}{1-x} \in [0, 1)$; concavity gives $\overline{v}^{cv}_{G_j}(y) \geq \lambda\, \overline{v}^{cv}_{G_j}(x) + (1 - \lambda)\, \overline{v}^{cv}_{G_j}(1) \geq \overline{v}^{cv}_{G_j}(x)$, strictly in the second case.

Step 2: monotonicity of u_i. Let $p_i < p_i'$, and let G^*_j be agent j's equilibrium signal for priors (p_i', p_j). By Step 1,

$$u_i(p_i, p_j) \leq \overline{v}^{cv}_{G^*_j}(p_i) \leq (<)\ \overline{v}^{cv}_{G^*_j}(p_i') = u_i(p_i', p_j).$$

Finally, since $v(x) + v(-x) = 1$, $u_i(p_i, p_j) = 1 - u_j(p_j, p_i)$, so, by the above applied to agent j, u_i is non-increasing (strictly decreasing) in p_j. □

OA.1.3 Informativeness and selection reversals

This subsection studies how informativeness in the symmetric equilibrium of Theorem 1 responds to the prior, and how often the firm selects the agent with the lower posterior.

For a fixed p, a higher a means a more informative bad news signal. In equilibrium, a rises with p, but this does not imply a more informative signal: if a rises only a little relative to p, the signal may become less informative. I view signals as experiments $\pi_i : \{0,1\} \to [0,1]$ in the Blackwell order (Blackwell, 1953).

A bad news experiment sends $s_i = 0$ or $s_i = a$ if $\theta_i = 0$, and always $s_i = a$ if $\theta_i = 1$, so it is characterized by the conditional probability of revealing the bad type, $\frac{\mathbb{P}(s_i=0)}{\mathbb{P}(\theta_i=0)}$. A higher probability gives a more informative experiment, since a lower one is obtained by garbling. Hence, I say that agent *i reveals more for* p_{i1} *than for* p_{i2} if $\frac{\mathbb{P}(s_{i1}=0)}{1-p_{i1}} > \frac{\mathbb{P}(s_{i2}=0)}{1-p_{i2}}$ for the corresponding equilibrium strategies G_{i1}, G_{i2}. By Theorem 1, this measure depends on a single variable:

$$\mathbb{P}(s_i = 0) = \frac{1}{\mathcal{E}_{\tilde{v}}(a)} \text{ and } p = a\left(1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)}\right) \text{ hence } \frac{\mathbb{P}(s_i = 0)}{1-p} = \frac{1}{1 + (\mathcal{E}_{\tilde{v}}(a) - 1)(1-a)}.$$

This gives the following lemma.

Lemma OA.3. *Let $v(x)$ be well S-shaped, $p < \overline{p}$, and $\{0, a\}$ the equilibrium signal. Informativeness strictly increases in the prior at p if and only if $(\mathcal{E}_{\tilde{v}}(x) - 1)(1-x)$ strictly decreases at $x = a$.*

Proof. By Theorem 1, $\frac{\mathbb{P}(s_i=0)}{1-p} = \frac{1}{1+(\mathcal{E}_{\tilde{v}}(a)-1)(1-a)}$, where, by its proof, $a(p)$ is continuous and strictly increasing, so it maps $(p, p+\varepsilon)$ and $(p-\varepsilon, p)$ onto a right and a left neighbourhood of $a(p)$. Therefore agents reveal more for all $p_h \in (p, p+\varepsilon)$ and less for all $p_l \in (p-\varepsilon, p)$ if and only if $(\mathcal{E}_{\tilde{v}}(x) - 1)(1-x)$ is strictly decreasing at $x = a$. □

Corollary OA.1. *Let $v(x)$ be well S-shaped with $\mathcal{E}_{\tilde{v}}(x)$ non-increasing. Then in equilibrium agents reveal more for p_2 than for p_1 whenever $0 < p_1 < p_2 \leq \overline{p}$.*

Proof. By Theorem 1, $\frac{\mathbb{P}(s_i=0)}{1-p} = \frac{1}{1+(\mathcal{E}_{\tilde{v}}(a)-1)(1-a)}$, where $a = a(p)$ is strictly increasing on $(0, \overline{p}]$ with $a(\overline{p}) = 1$. If $\mathcal{E}_{\tilde{v}}(x)$ is non-increasing, then, since $\mathcal{E}_{\tilde{v}}(x) - 1 \geq 1$ by Lemma 3, $(\mathcal{E}_{\tilde{v}}(x) - 1)(1-x)$ is strictly decreasing, so this ratio is strictly increasing in p. □

Some algebra shows that Corollary OA.1 applies to the natural examples.

Example OA.3. Numerically, for $v(x)$ generated by i.i.d. Gumbel or normal shocks with $\mathrm{Var}[\varepsilon_i] \geq \overline{V}$, $\mathcal{E}_{\tilde{v}}(x)$ is non-increasing, so agents reveal more as p increases.

However, the elasticity can vary flexibly, and better agents may then reveal less.

Example OA.4. There exists a well S-shaped $v(x)$ with $\tilde{v}(x) = -\frac{x^2}{16(1-x)^2}$ on $[0, \frac{1}{2}]$ such that in equilibrium agents reveal less for p_2 than for p_1 if $0 < p_1 < p_2 \leq 0.375$.

Selection reversals. Consider how often noise makes the firm select the agent with the lower posterior in the symmetric equilibrium. For $p < \overline{p}$, posteriors differ with probability $\frac{2}{\mathcal{E}_{\tilde{v}}(a)}(1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)})$, and the firm then selects the agent with $s_i = 0$ with probability $1 - v(a)$. If $\mathcal{E}_{\tilde{v}}(x)$ is non-decreasing, the first factor does not increase in p, as $\frac{1}{\mathcal{E}_{\tilde{v}}(a)} \leq \frac{1}{2}$, and the second decreases, so reversals become rarer. Under the condition of Corollary OA.1, which makes informativeness increase in p, the two factors move in opposite directions, and either can dominate. For the two classes from Example 1, which satisfy it by Example OA.3, a direct computation shows that they still become rarer, but for $v(x) = \frac{1}{2} + 0.6x - 0.1e^{5-\frac{5}{x}}$ on $[0, 1]$, the probability of a reversal increases in p for $p < 0.36$.

OA.2. Designing the Selection Rule

Suppose that signals are costly: agent i pays $\kappa\, c(G_i)$, where $\kappa > 0$ and $c(G_i) = \int_0^1 \phi\, dG_i - \phi(p_i)$ for a strictly convex ϕ, continuous on $[0, 1]$, so that his payoff is $\mathbb{E}[v(s_i - s_j)] - \kappa\, c(G_i)$. The firm does not bear the cost, and her utility is unchanged.

Lemma OA.4. *For any priors and selection rule, equilibria exist and form a product set.*

Proof. Since agent i's cost does not depend on G_j, both agents have the same best replies as in the zero-sum game in which agent 1 receives $W(G_1, G_2) := \mathbb{E}[v(s_1 - s_2)] - \kappa c(G_1) + \kappa c(G_2)$ and agent 2 receives $-W(G_1, G_2)$. W is affine in each G_i and continuous in the weak* topology, as v and ϕ are continuous, and each $\mathcal{G}(p_i)$ is convex and weak* compact. By Sion's minimax theorem, this game has a value and optimal strategies, and its equilibria are exactly the pairs of optimal strategies. □

Unlike the equilibrium value of this zero-sum game, the agents' own payoffs need not be the same in all equilibria.

OA.2.1 The variance cost

Let $\phi(x) = x^2$, so that $c(G_i) = \mathrm{Var}_{G_i}[s_i]$ is the expected reduction in the residual variance of θ_i. As the linear part of $v(x)$ adds only a linear function of s_i to agent i's value function, best replies depend on $v(x)$ only through $\tilde{v}(x)$ and

$$\lambda := \frac{\kappa}{|v(1) - \frac{1}{2} - v'(0)|},$$

and the normalized value function against G_j is $f(s_i) := \overline{v}(s_i) - \lambda s_i^2$.

Proposition OA.2. *Let $v(x)$ be well S-shaped, the priors (p, p), and $c(G_i) = Var_{G_i}[s_i]$.*

(i) If $p \geq \overline{p}_\lambda := 1 - \frac{1-\lambda}{\mathcal{E}_{\tilde{v}}(1)}$, the unique equilibrium is $(\{0,1\};\{0,1\})$.

(ii) If $\lambda \geq \frac{|\tilde{v}(p)|}{p^2}$, the unique equilibrium is $(\{p\};\{p\})$.

(iii) Otherwise, the unique equilibrium is $(\{0,a\};\{0,a\})$, where $a \in (p,1)$ solves $p = a - \frac{|\tilde{v}(a)|-\lambda a^2}{|\tilde{v}'(a)|}$, and $\mathbb{P}(s_i = 0) = \frac{1}{\mathcal{E}_{\tilde{v}}(a)}\left(1 - \frac{\lambda a^2}{|\tilde{v}(a)|}\right)$.

The conditions in (i) and (ii) are mutually exclusive.

Proof. Exclusivity. By the concavity of $\tilde{v}(x)$ on $[0,1]$ (property 2 in Appendix A.1), $\tilde{v}(p) \leq \tilde{v}(1) + (p-1)\tilde{v}'(1) = -1 + (1-p)\mathcal{E}_{\tilde{v}}(1)$. So if (i) holds, $\lambda \leq 1 - (1-p)\mathcal{E}_{\tilde{v}}(1) \leq |\tilde{v}(p)| < \frac{|\tilde{v}(p)|}{p^2}$.

Reduction to Lemma 4. Let agent j play $\{0,a\}$ with $a \in (p,1]$, and let $\gamma := \frac{a-p}{a} \in (0,1)$. Agent i's normalized payoff is the expectation of

$$f(s) = \gamma\tilde{v}(s) + (1-\gamma)\tilde{v}(s-a) - \lambda s^2,$$

which differs from the value function $\overline{v}$ in the proof of Lemma 4 only by $-\lambda s^2$. The parts of that proof headed *Uniqueness*, *Best reply to* $\{0,a\}$, *Step 1*, *Step 2* and *(i)* carry over verbatim, with $E \times E$ now given by Lemma OA.4 and symmetry, $\overline{v}$ replaced by f, l by the line through $(0, f(0))$ and $(a, f(a))$, whose slope is $\frac{\tilde{v}(a)-\lambda a^2}{a}$, $d := l - f$, and w in Step 2 by $\int_0^a \tilde{v}(s-t)dG(t) - \lambda s^2$: the cost adds $2\lambda s$ to $d'(s)$, which keeps d' convex, and $-2\lambda s$ to $f'(s)$, which keeps $f'(s) < f'(a)$ on $(a,1]$. Only four formulas change:

(a) As $f'(a) = \gamma\tilde{v}'(a) - 2\lambda a$ and $\gamma a = a - p$,

$$d'(a) = 0 \;\Leftrightarrow\; \gamma = \frac{1}{\mathcal{E}_{\tilde{v}}(a)}\left(1 - \frac{\lambda a^2}{|\tilde{v}(a)|}\right) \;\Leftrightarrow\; p = a - \frac{|\tilde{v}(a)| - \lambda a^2}{|\tilde{v}'(a)|}.$$

(b) In Step 1, $d' \equiv 0$ with $\tilde{v}'(s) = -Ks$ on $[0,a]$ requires $K(1-2\gamma) = 2\lambda$, i.e., $p = a\left(\frac{1}{2} + \frac{\lambda}{K}\right)$.

(c) In Step 2, $w'(a) = -K(a-p) - 2\lambda a$ and $w(a) = -\frac{K}{2}\mathbb{E}_G[(a-t)^2] - \lambda a^2$, so the secant slope exceeds $w'(a)$ by $\frac{K}{a}\left(\frac{a^2}{2} - \mathbb{E}_G[t^2]\right) + \lambda a > \frac{K}{a}\left(\frac{a^2}{2} - ap\right) + \lambda a = 0$, by (b).

(d) For $a = 1$, $\gamma = 1 - p$ and $d'(1) = -1 + \lambda + (1-p)\mathcal{E}_{\tilde{v}}(1)$, so $d'(1) \leq 0$ if and only if $p \geq \overline{p}_\lambda$.

In particular, (i) follows from (d) as in the proof of Lemma 4(i), and, whenever $d'(a) = 0$ for some $a \in (p, 1)$, $(\{0, a\}; \{0, a\})$ is the unique equilibrium.

(ii) Against $\{p\}$, $f(s) = \tilde{v}(s-p) - \lambda s^2$, and $f'(p) = -2\lambda p$. So $\{p\}$ is a best reply if and only if f lies below its tangent at p, i.e., if and only if $\tilde{v}(s-p) \leq \lambda(s-p)^2$ on $[0, 1]$. For $s > p$, this holds strictly, as $\tilde{v} < 0$ on $(0, 1]$. For $s < p$, with $t = p - s$, it reads $\frac{|\tilde{v}(t)|}{t^2} \leq \lambda$, and as this ratio is non-decreasing on $(0, 1]$ by Lemma 3, it holds for all $t \in (0, p]$ if and only if $\lambda \geq \frac{|\tilde{v}(p)|}{p^2}$. Then every best reply to $\{p\}$ is supported on $[0, p]$ and has mean p, so it is $\{p\}$, and the equilibrium is unique.

(iii) Let $p < \overline{p}_\lambda$ and $\lambda < r(p)$, where $r(t) := \frac{|\tilde{v}(t)|}{t^2}$ is continuous and non-decreasing on $(0, 1]$. Let $h(a) := a - \frac{|\tilde{v}(a)| - \lambda a^2}{|\tilde{v}'(a)|} = a\big(1 - \frac{1-\lambda/r(a)}{\mathcal{E}_{\tilde{v}}(a)}\big)$, continuous on $(0, 1]$, with $h(1) = \overline{p}_\lambda > p$. Let $a_0 := \sup\{t \in (0, 1] : r(t) \leq \lambda\}$, and $a_0 := 0$ if this set is empty; by continuity and monotonicity of r, $a_0 < p$. If $a_0 > 0$, then $r(a_0) = \lambda$ and $h(a_0) = a_0 < p$. If $a_0 = 0$, then $\lambda < r(a)$ for all $a \in (0, 1]$, so $h(a) \in [\frac{a}{2}, a)$ by Lemma 3, and $h(a) \to 0$ as $a \to 0$. In both cases, by continuity, $h(a) = p$ for some $a \in (a_0, 1)$. As $r(a) > \lambda$ there, $p = h(a) < a$, so $\gamma \in (0, 1)$, and $d'(a) = 0$ by (a). Hence $(\{0, a\}; \{0, a\})$ is the unique equilibrium, with $\mathbb{P}(s_i = 0) = \gamma$ as stated. □

For $\lambda = 0$, (ii) never applies, and Proposition OA.2 reduces to Theorem 1. Since λ is the cost relative to the curvature of $v(x)$, noise that flattens $v(x)$ makes the same cost weigh more. For instance, for $\varepsilon_i \sim U[0, u]$ with $u \geq 1$, $\tilde{v}(x) = -x|x|$ and $\lambda = 2\kappa u^2$: agents reveal fully if $p \geq \frac{1+\lambda}{2}$, pool if $\lambda \geq 1$, and otherwise play $\{0, \frac{2p}{1+\lambda}\}$ with $\mathbb{P}(s_i = 0) = \frac{1-\lambda}{2}$. Without the cost, informativeness does not depend on $u \geq 1$; with it, it decreases in u.

OA.2.2 Robust implementation

For the rules of Section 4, let ϕ be continuously differentiable on $[0, 1]$, including the endpoints, and let $\psi(s) := (1-s)\phi(0) + s\phi(1) - \phi(s)$, which is positive on $(0, 1)$ by strict convexity, with $\lim_{s\to 0} \frac{\psi(s)}{s} = \phi(1) - \phi(0) - \phi'(0) > 0$ and $\lim_{s\to 1} \frac{\psi(s)}{1-s} = \phi'(1) - \phi(1) + \phi(0) > 0$.

Proposition OA.3. *Let ϕ be continuously differentiable on $[0, 1]$.*

(i) If $v(x) = \frac{1}{2} + ax$, the unique equilibrium is $(\{p_1\}; \{p_2\})$ for all (p_1, p_2).

(ii) Let $v(x)$ be strictly positively [negatively] balanced with $\liminf_{s\to 0_+} \frac{|g(s)|}{s} > 0$. Then $m := \inf_{s\in(0,1)} \frac{|g(s)|}{\psi(s)} > 0$, and full revelation is the unique equilibrium for all (p_1, p_2) with $p_1, p_2 > \frac{1}{2} + \frac{\kappa}{2m}$ $\big[p_1, p_2 < \frac{1}{2} - \frac{\kappa}{2m}\big]$.

(iii) Let $v(x)$ be positively or negatively balanced with $\liminf_{s\to 0_+} \frac{|g(s)|}{s} = 0$. Then $(\{0,1\};\{0,1\})$ is not an equilibrium for any (p_1, p_2).

Proof. (i) Agent i's payoff is $\frac{1}{2} + a(p_i - p_j) - \kappa c(G_i)$, and by strict convexity, $c(G_i) > 0$ unless $G_i = \{p_i\}$.

(ii)–(iii) Let agent j play $\{0,1\}$. With $\hat{v}(x) = ax + g(x)$ on $[0,1]$ and $g(x) = g(1-x)$, agent i's value function is, up to a constant,

$$(1-p_j)\hat{v}(s) - p_j\hat{v}(1-s) = a(s-p_j) + (1-2p_j)g(s),$$

and he maximizes the expectation of this function minus $\kappa\phi(s)$. By Kamenica and Gentzkow (2011, Corollary 2), $\{0,1\}$ is a best reply if and only if this function lies below its chord on $[0,1]$. As $g(0) = g(1) = 0$, this reads

$$(2p_j - 1)g(s) \geq \kappa\psi(s) \quad \forall s \in (0,1), \tag{FR}$$

which does not depend on p_i; if (FR) is strict on $(0,1)$, $\{0,1\}$ is the unique best reply.

(ii) Consider the positive case. The ratio $\frac{g}{\psi}$ is continuous and positive on $(0,1)$, and $\liminf_{s\to 0} \frac{g(s)}{\psi(s)} > 0$ by the limit of $\frac{\psi(s)}{s}$ above. As $g(s) = g(1-s)$, $\liminf_{s\to 1} \frac{g(s)}{1-s} > 0$ too, so $\liminf_{s\to 1} \frac{g(s)}{\psi(s)} > 0$, and hence $m > 0$. If $p_j > \frac{1}{2} + \frac{\kappa}{2m}$, then $(2p_j - 1)g(s) \geq (2p_j - 1)m\psi(s) > \kappa\psi(s)$ on $(0,1)$, so (FR) is strict. Hence, if both priors exceed $\frac{1}{2} + \frac{\kappa}{2m}$, full revelation is each agent's unique best reply to itself, and by Lemma OA.4, it is the unique equilibrium. The negative case is symmetric.

(iii) Consider the positive case. If $p_j \leq \frac{1}{2}$, the left side of (FR) is non-positive, and the right side is positive. If $p_j > \frac{1}{2}$, take $s_n \to 0$ with $\frac{g(s_n)}{s_n} \to 0$; then $(2p_j - 1)\frac{g(s_n)}{s_n} \to 0$, while $\kappa\frac{\psi(s_n)}{s_n} \to \kappa\big(\phi(1) - \phi(0) - \phi'(0)\big) > 0$, so (FR) fails. The negative case is symmetric. □

For the variance cost, $\psi(s) = s(1-s)$; for $v^*(x)$, $g(s) = \frac{s(1-s)}{2}$, so $m = \frac{1}{2}$ and (ii) gives full revelation for $p > \frac{1}{2} + \kappa$, in line with Proposition OA.2(i), as $\lambda = 2\kappa$. If instead $\phi'(0) = -\infty$, as for entropy, then $\frac{\psi(s)}{s} \to \infty$ as $s \to 0$, while $\frac{|g(s)|}{s} \leq a$ for every balanced rule, so by (FR), no balanced rule implements full revelation for any $\kappa > 0$.

OA.2.3 Triangular rules: the remaining case

This subsection completes the construction of Appendix B.4 for the triangular selection rules of Subsection 4.2 and uses its notation, proper u-partitions included. Binary support survives a slight violation of well S-shapedness for extreme priors, $p \leq \frac{1}{2}u$

or $p \geq 1 - \frac{1}{2}u$, but fails for all $p \in (\frac{1}{2}u, 1 - \frac{1}{2}u)$. The case left open in the proof of Proposition 13, in which the solution a of the equations in its case (iii) satisfies $a > 1-u$, shows how complex equilibria can be for general $v(x)$: as the signal 1 must be treated separately, an equilibrium may require a three-way mixture of two proper u-partitions and a mass point.

Proposition OA.4. *Let $v(x)$ be the CDF of a triangular distribution on $[-u, u]$, let $p > \frac{1}{2}$ and $u < 2 - 2p$, and let a, the solution of the equations in case (iii) of the full statement in the proof of Proposition 13, satisfy $a > 1 - u$. Then a symmetric equilibrium signal G is constructed as follows: let $k = \lceil \frac{1}{u} \rceil$, and let $\tilde{G}$ be a $(\frac{k-1}{k}, \frac{1}{k})$ mix of the proper u-partition of $[0, 1-u]$ and $\delta_{\{1\}}$. Then let G be the*

$$\left(\frac{(2p-a)k}{(k-1)(1-u)+2-ka}, \frac{(k-1)(1-u)+2-2pk}{(k-1)(1-u)+2-ka} \right)$$

mix of $\tilde{G}$ and the proper u-partition of $[0, a]$, where $a \in (1-u, (k-1)u)$ is the unique solution of $(1-a)\overline{v}(0) + a\overline{v}(1) = \overline{v}(a)$.

Proof. Throughout, (i) and (iii) refer to the cases of the full statement in the proof of Proposition 13, whose notation I use. Notice that G is a feasible distribution, since the mean of $\tilde{G}$ is $\frac{k-1}{k}\frac{1-u}{2} + \frac{1}{k} = \frac{Q}{2k}$ with $Q := (k-1)(1-u)+2$, the mean of the proper u-partition of $[0, a]$ is $\frac{a}{2}$, and $\beta := \frac{(2p-a)k}{Q-ka}$ solves $\beta\frac{Q}{2k} + (1-\beta)\frac{a}{2} = p$.

Suppose that a from (iii) satisfies $a > 1-u$, and let $k = \lceil \frac{1}{u} \rceil \geq 2$, so that $(k-1)u < 1$ and $\lceil \frac{1-u}{u} \rceil = k-1$. I first show that against $\tilde{G}$, the $(\frac{k-1}{k}, \frac{1}{k})$ mix of the proper u-partition χ' of $[0, 1-u]$ and $\delta_{\{1\}}$, $\overline{v}'$ is constant on $[0, (k-1)u]$ and increasing and affine on $[(k-1)u, 1]$. I use Lemma 7 and the masses in its proof, with $k-1$ in place of k: its top atom is $1-u$, of mass $\frac{1}{k}$, its largest atom of the form iu is $(k-2)u$, of mass $\frac{1}{k(k-1)}$, and its other atoms are at most $1-2u$. On $[0, 1-u]$, the χ' part of $\overline{v}'$ is constant, and the $\delta_{\{1\}}$ part, $\frac{1}{k}\frac{(x+u-1)^+}{u^2}$, vanishes. For $x \in (1-u, 1)$, $(x, x+u)$ contains only the atom 1; $(x-u, x)$, which lies in $(1-2u, 1)$, contains exactly the atoms $1-u$ and $(k-2)u$ of $\tilde{G}$ if $x < (k-1)u$, of total mass $\frac{k-1}{k}\left(\frac{1}{k} + \frac{1}{k(k-1)}\right) = \frac{1}{k}$, and only $1-u$, of mass $\frac{k-1}{k^2}$, if $x > (k-1)u$. So $\overline{v}'' = 0$ on $(1-u, (k-1)u)$ and $\overline{v}'' = \frac{1}{k^2u^2}$ on $((k-1)u, 1)$, outside a finite set. By Lemma 7, the constant value is $\overline{v}'(0) = \frac{k-1}{k} \cdot \frac{2(k-1)u-(1-u)}{u^2k(k-1)} = \frac{(2k-1)u-1}{u^2k^2}$.

So against $\tilde{G}$, $\overline{v}'(x)$ is constant on $[0, (k-1)u]$ and increasing linear on $[(k-1)u, 1]$, while against the proper u-partition of $[0, a]$ it is constant on $[0, a]$ and convex on $[a, 1]$ by Lemma 7. Hence, if the other agent plays G, $\overline{v}(x)$ is linear on $[0, a]$ and $\overline{v}'(x)$ is convex on $[a, 1]$, and as in (iii), if a solves $(1-a)\overline{v}(0) + a\overline{v}(1) = \overline{v}(a)$, then $[0, a] \cup \{1\}$ is

the concavification support, and (G, G) is an equilibrium.

It remains to show that such an $a \in (1-u, (k-1)u)$ exists and is unique, and that $\beta \in (0,1)$. Let a_3 and $k_3 = \lceil \frac{2-2p}{u} \rceil - 1$ be a and k from (iii). Since $a_3 \le k_3 u$ and $k_3 \le k-1$, $a_3 > 1-u > (k-2)u$ requires $k_3 = k-1$, and then $a_3 > 1-u$ is equivalent to $p < p_u := 1 - \frac{k(k-1)u^2}{2((2k-1)u-1)}$. Let χ be the proper u-partition of $[0,a]$ and, for a signal G' of the other agent, let $s_{G'}$ be the slope of $\overline{v}(x)$ on $[0,a]$ and $C_{G'} := \overline{v}(1) - \overline{v}(0)$, so that the condition on a is $s_G = C_G$. A direct computation gives

$$s_{\tilde{G}} - C_{\tilde{G}} = -\frac{((k-1)u-1)^2}{2k^2u^2}, \qquad \frac{d}{da}(s_\chi - C_\chi) = -\frac{1-a}{(k-1)u^2},$$

and $s_\chi = \frac{2(k-1)u-a}{u^2k(k-1)}$ by Lemma 7. Hence

$$\Lambda(a) := (Q-ka)(s_G - C_G) = -(2p-a)\frac{((k-1)u-1)^2}{2ku^2} + (Q-2kp)(s_\chi - C_\chi)$$

has the sign of $s_G - C_G$, as $Q - ka \ge (k+1)(1-(k-1)u) > 0$, and $\Lambda(a)$ is quadratic in a with the leading coefficient $\frac{Q-2kp}{2(k-1)u^2}$. Since $p < p_u$ and $\frac{Q}{2k} - p_u = \frac{(k-1)((k-1)u-1)^2}{2k((2k-1)u-1)} > 0$, $2kp < Q$ and Λ is strictly convex. Moreover,

$$\Lambda(1-u) = \frac{(1+u)((2k-1)u-1)}{k(k-1)u^2}(p_u - p) > 0,$$
$$\Lambda((k-1)u) = -\frac{(k+1)(2p-1)((k-1)u-1)^2}{2ku^2} < 0,$$

so Λ has a unique root in $(1-u, (k-1)u)$. And $a < 1 < 2p$ and $2kp < Q$ give $\beta \in (0,1)$. □

OA.2.4 Unequal priors

Theorem 2 and the comparison with the noiseless rule in Subsection 4.1 concern equal priors. Let now $p_2 < p_1$, and let $u_F^* := p_1 + p_2 - p_1p_2$ be the first best. If $p_2 > \frac{1}{2}$, $v^*(x)$, for which $\overline{p} = \frac{1}{2}$, makes both agents reveal fully (Proposition 1) and attains u_F^*, as it does at $p_2 = \frac{1}{2}$ in one of its equilibria (Remark 1), while under the noiseless rule agents never reveal fully. So let $p_2 < \frac{1}{2}$. Then no well S-shaped rule attains the first best: its density is positive on $(-1,1)$, so $v(x) < 1$ for $x < 1$, and by the necessity part of Appendix B.11 both agents would have to reveal fully, which requires $p_2 \ge \overline{p} \ge \frac{1}{2}$. As in the proof of Proposition OA.1, $u_F = \frac{p_1+p_2}{2} + \mathbb{E}[\Psi(s_1 - s_2)]$ with $\Psi(x) = x\hat{v}(x)$; for

$v^*(x)$, $\Psi^*(x) = x^2 - \frac{|x|^3}{2} = |x|T(|x|)$ with $T(x) := x - \frac{x^2}{2}$.

The rule $v^*(x)$. As the density of $v^*(x)$ is linear on $[0,1]$, Proposition 2 does not apply, but the equilibria are explicit.

Remark OA.2. Let $v(x) = v^*(x)$ and $p_2 < \min(p_1, \frac{1}{2})$.

(i) If $p_2 < \frac{p_1}{\sqrt{2}}$, the unique equilibrium is $(\{p_1\}; \{0, b\})$ with $b = \min(\sqrt{2}p_1, 1)$, and $u_F = \frac{p_1+p_2}{2} + \left(1 - \frac{p_2}{b}\right)\Psi^*(p_1) + \frac{p_2}{b}\Psi^*(b - p_1)$.

(ii) If $p_2 \geq \frac{p_1}{\sqrt{2}}$, agent 2 plays $\{0, 2p_2\}$ in every equilibrium, and agent 1 plays any signal supported on $[0, 2p_2]$ with $\mathbb{E}[s_1^2] = 2p_2^2$, such as $\{0, \frac{2p_2^2}{p_1}\}$. In every equilibrium, $u_F = \frac{p_1+p_2}{2} + 4p_2^2 - 2p_1p_2 - 5p_2^3 + 3p_1p_2^2$.

Proof. Let agent j play a signal G_j supported on $[0, c]$. With $\tilde{v}(x) = -x|x|$, agent i's normalized value function $\overline{v}(s) = \mathbb{E}[\tilde{v}(s - s_j)]$ has $\overline{v}''(s) = 2 - 2G_j(s^-) - 2G_j(s)$, which is non-increasing, so $\overline{v}(s)$ is S-shaped as in Lemma 1. On $[c, 1]$, $\overline{v}(s) = -\mathbb{E}[(s - s_j)^2]$ is strictly concave, and $N(s) := s\overline{v}'(s) - \overline{v}(s) + \overline{v}(0) = 2\mathbb{E}[s_j^2] - s^2$. If $G_j = \{0, c\}$ with weight $q = \frac{p_j}{c}$ on c, then $\overline{v}'' = 2(2q - 1)$ on $(0, c)$. So agent i's best reply to $\{0, c\}$ is: $\{p_i\}$, uniquely, if $q < \frac{1}{2}$; uniquely $\{0, b\}$ with $b = \min(\sqrt{2p_jc}, 1)$ if $q > \frac{1}{2}$ and $p_i < b$, and $\{p_i\}$ if $q > \frac{1}{2}$ and $p_i \geq b$; and, if $q = \frac{1}{2}$ and $p_i \leq c$, every signal supported on $[0, c]$, as $\overline{v}(s)$ is linear on $[0, c]$.

(i) Against $\{p_1\}$, so $c = p_1$ and $q = 1$, agent 2's unique best reply is $\{0, b\}$, as $p_2 < p_1 < b$. Its weight $\frac{p_2}{b}$ on b is below $\frac{1}{2}$, as $p_2 < \frac{p_1}{\sqrt{2}}$ and $p_2 < \frac{1}{2}$, so agent 1's unique best reply is $\{p_1\}$. As the equilibria form a product set, the equilibrium is unique, and u_F follows from $s_1 - s_2 \in \{p_1, p_1 - b\}$.

(ii) Let $a_0 := \frac{2p_2^2}{p_1}$, so that $p_1 \leq a_0 < 2p_2 < 1$. Against $\{0, a_0\}$, $q = \frac{p_1^2}{2p_2^2} > \frac{1}{2}$ and $\sqrt{2p_1a_0} = 2p_2$, so agent 2's unique best reply is $\{0, 2p_2\}$; against $\{0, 2p_2\}$, $q = \frac{1}{2}$ and $p_1 \leq \sqrt{2}p_2 < 2p_2$, so $\{0, a_0\}$ is a best reply. As the equilibria form a product set, agent 2 plays $\{0, 2p_2\}$ in every equilibrium, and agent 1's equilibrium signals are the signals G_1 supported on $[0, 2p_2]$ to which $\{0, 2p_2\}$ is a best reply. As agent 2's value function against such G_1 is S-shaped and concave on $[2p_2, 1]$, this holds if and only if $N(2p_2) = 0$, i.e. $\mathbb{E}[s_1^2] = 2p_2^2$. For such G_1, $\mathbb{E}[(s_1 - s_2)^2] = 4p_2^2 - 2p_1p_2$ and, as $s_2 \in \{0, 2p_2\}$ with equal weights, $\mathbb{E}\left[|s_1 - s_2|^3 \,\middle|\, s_1\right] = \frac{1}{2}\left(s_1^3 + (2p_2 - s_1)^3\right) = 4p_2^3 - 6p_2^2s_1 + 3p_2s_1^2$, so $\mathbb{E}[|s_1 - s_2|^3] = 10p_2^3 - 6p_1p_2^2$, which gives u_F. □

Case (ii) extends Remark OA.1, and at $p_1 = p_2$ it gives the bound of Theorem 2.

The noiseless rule. Let G_p be the noiseless equilibrium signal at prior p (Appendix B.1) and $\lambda := \frac{p_2}{p_1}$. Then the unique equilibrium (Boleslavsky and Cotton, 2015, Lemma 3) is $(G_{p_1}; (1-\lambda)\delta_0 + \lambda G_{p_1})$: the favorite plays as with equal priors, and the underdog plays the same signal or reveals bad news. Indeed, with a signal s an agent wins with probability $\overline{G}_j(s) := G_j(s^-) + \frac{1}{2}\big(G_j(s) - G_j(s^-)\big)$. Since $\overline{G}_{p_1}(s) \le \frac{s}{2p_1}$ on $[0,1]$, with equality on $\{0\} \cup \operatorname{supp} G_{p_1}$, every signal of agent 2 wins with probability at most $\frac{p_2}{2p_1}$, attained by every signal supported on $\{0\} \cup \operatorname{supp} G_{p_1}$. Against agent 2's signal, a positive signal s of agent 1 wins with probability at most $1 - \lambda + \frac{\lambda s}{2p_1}$, with equality on $\operatorname{supp} G_{p_1}$, which has no atom at 0, and the signal 0 wins with probability $\frac{1-\lambda}{2} < 1 - \lambda$; so every signal of agent 1 wins with probability at most $1 - \lambda + \frac{\lambda \mathbb{E}[s_1]}{2p_1} = 1 - \frac{\lambda}{2}$, which G_{p_1} attains. As $u_F = \mathbb{E}[\max(s_1, s_2)]$,

$$u_F^0 = p_1 + \lambda\big(u_F^0(p_1) - p_1\big) = \begin{cases} p_1 + \frac{p_2}{3}, & p_1 \le \frac{1}{2}, \\ p_1 + \frac{p_2(1-p_1)(7p_1^2 - 5p_1 + 1)}{3p_1^3}, & p_1 > \frac{1}{2}, \end{cases}$$

with $u_F^0(p)$ from Appendix B.1. Its share of the first best exceeds $\frac{2}{3}$: for $p_1 \le \frac{1}{2}$, $u_F^0 \ge \frac{2}{3}(p_1 + p_2) > \frac{2}{3}u_F^*$, and for $p_1 > \frac{1}{2}$, $u_F^0 \ge p_1$ and $u_F^* \le p_1(2 - p_1)$, so the share is at least $\frac{1}{2-p_1} > \frac{2}{3}$.

Remark OA.3. Let $p_2 < \min(p_1, \frac{1}{2})$. Then $v^*(x)$ gives the firm more than the noiseless rule if and only if $p_2 > \frac{3-\sqrt{3}}{6}$ and $p_1 < \overline{\phi}(p_2)$, where $\overline{\phi}(p_2) := \frac{p_2(1+24p_2-30p_2^2)}{3(1+4p_2-6p_2^2)}$ if this is at most $\frac{1}{2}$, and otherwise $\overline{\phi}(p_2)$ is the root in $(\frac{1}{2}, \sqrt{2}p_2)$ of

$$p_1^4\big(18p_2^2 - 12p_2 - 3\big) + p_1^3\big(17p_2 + 24p_2^2 - 30p_2^3\big) - 24p_1^2p_2 + 12p_1p_2 - 2p_2 = 0.$$

In particular, the noiseless rule is better whenever the favorite pools under $v^*(x)$.

Proof. Let $D := u_F - u_F^0$. In case (i) of Remark OA.2, both payoffs are linear in p_2 for a fixed p_1. At $p_2 = 0$, $D = \Psi^*(p_1) - \frac{p_1}{2} < 0$, as $T(x) < \frac{1}{2}$ for $x < 1$. At the other end of case (i), $D < 0$ as well: if $p_1 > \frac{1}{\sqrt{2}}$, then $b = 1$ and at $p_2 = \frac{1}{2}$,

$$D = \frac{(p_1 - 1)(3p_1^4 - 6p_1^3 + 14p_1^2 - 10p_1 + 2)}{12p_1^3},$$

whose quartic has no root in $[\frac{1}{\sqrt{2}}, 1]$ and is positive there; if $p_1 \le \frac{1}{\sqrt{2}}$, the end is $p_2 = \frac{p_1}{\sqrt{2}}$, where the two cases of Remark OA.2 give the same payoff, and $D < 0$ there by the next

step. So $D < 0$ in case (i). In case (ii), $\partial_{p_1} u_F = \frac{1}{2} - 2p_2 + 3p_2^2$, while $\partial_{p_1} u_F^0 = 1$ for $p_1 < \frac{1}{2}$ and $\partial_{p_1} u_F^0 = 1 - \frac{p_2(2p_1-1)^2}{p_1^4} \geq 1 - p_2$ for $p_1 > \frac{1}{2}$, as $\frac{(2p_1-1)^2}{p_1^4}$ increases to 1 on $[\frac{1}{2}, 1]$. So $\partial_{p_1} D \leq -\frac{1}{2} - p_2 + 3p_2^2 < 0$, and D is strictly decreasing in p_1 on $[p_2, \sqrt{2}p_2]$. At $p_1 = p_2$, $D = p_2 + 2p_2^2(1-p_2) - \frac{4p_2}{3}$, which is positive if and only if $6p_2^2 - 6p_2 + 1 < 0$, i.e. $p_2 > \frac{3-\sqrt{3}}{6}$. At $p_1 = \sqrt{2}p_2$, $D < 0$: for $p_1 \leq \frac{1}{2}$, $D = \frac{p_2}{6} + 4p_2^2 - 5p_2^3 - p_1\left(\frac{1}{2} + 2p_2 - 3p_2^2\right)$ decreases in p_1 and vanishes at $p_1 = \frac{p_2(1+24p_2-30p_2^2)}{3(1+4p_2-6p_2^2)} < \sqrt{2}p_2$, as $1 + 24p - 30p^2 < 3\sqrt{2}(1 + 4p - 6p^2)$ for all p; for $p_1 \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, $12p_1^2 D$ at $p_2 = \frac{p_1}{\sqrt{2}}$ is a quintic in p_1 with no root in this interval, negative at $p_1 = \frac{1}{2}$. Hence D has a unique root in $(p_2, \sqrt{2}p_2)$ if $p_2 > \frac{3-\sqrt{3}}{6}$ and none otherwise. It is the formula if it is at most $\frac{1}{2}$, and for $p_1 > \frac{1}{2}$, $6p_1^3 D = 0$ is the quartic. □

The threshold $\overline{\phi}(p_2)$ is 0.223, 0.263, 0.331, 0.401, 0.472, 0.543 and 0.589 at $p_2 = 0.22$, 0.25, 0.3, 0.35, 0.4, 0.45 and 0.48, and tends to 0.625 as $p_2 \to \frac{1}{2}$; the formula applies for $p_2 \leq 0.420$. So with the underdog more likely bad than good, $v^*(x)$ beats the noiseless rule only on a thin band of close priors, about 3% of the triangle $\{p_2 < p_1\}$, and if $p_1 > \frac{1}{2}$, its payoff jumps up to the first best as p_2 crosses $\frac{1}{2}$.

Other well S-shaped rules. Other well S-shaped rules do not change this picture much. As in Step 2 of the proof of Proposition 2, let $w := -v''$ on $(0, 1]$, which is non-negative and non-decreasing, so that $\hat{v}(x) = v'(1)x + H_w(x)$ with $H_w(x) := \int_0^1 \min(t, x) w(t) dt$. As H_w is concave, $H_w(x) \geq x H_w(1)$, and as $\hat{v}(1) \leq \frac{1}{2}$, $\hat{v}(x) \leq \frac{\hat{v}(1)}{H_w(1)} H_w(x) \leq \frac{H_w(x)}{2H_w(1)}$. Replacing $\hat{v}$ by $\frac{H_w}{2H_w(1)}$ changes only the linear part and the scale of v, and hence does not change any equilibrium, and it raises Ψ, so without loss $\hat{v} = H_w$ with $\int_0^1 t w(t) dt = \frac{1}{2}$, i.e. $v(1) = 1$ and $v'(1) = 0$. For $c \in [0, 1)$, the step $w = \frac{\mathbf{1}\{t \geq c\}}{1-c^2}$ gives the *trapezoid rule* τ_c, with density $\frac{1-\max(|x|, c)}{1-c^2}$: flat on $[-c, c]$ and linear down to 0 at ± 1, so $\tau_0 = v^*$. It is generated by independent shocks uniform on intervals of lengths $1 + c$ and $1 - c$, but for $c \in (0, 1)$ by no i.i.d. shocks, as its characteristic function, a product of two sinc functions, is negative somewhere.

Remark OA.4. For any priors, the supremum of u_F over well S-shaped rules with $v'(x)$ not linear near 0 is the value of linear programs whose optimal rules are mixtures of at most three trapezoid rules, the linear rule included as the case $c = 1$, and it is approached by rules of the class close to these mixtures.

Proof. For a fixed pair of high signals (a, b) in Proposition 2, pooling and full revelation included, the conditions that make $(\{0, a\}; \{0, b\})$ the equilibrium are sign conditions on N in (N) at a and b, and A, B and $u_F = \frac{p_1+p_2}{2} + \mathbb{E}\left[|s_1 - s_2| \, H_w(|s_1 - s_2|)\right]$ are all linear in

w. Write $w(t) = \int \frac{\mathbf{1}\{t \geq c\}}{1-c^2}\, d\nu(c)$ for a probability measure ν on $[0,1]$, the normalization, where $c = 1$ stands for the linear rule. Then $u_F = \frac{p_1+p_2}{2} + \int U\, d\nu$, where $U(c)$ is the gain from τ_c, and the firm maximizes it subject to the two conditions, whose coefficients, like U, are continuous on $[0,1]$. If some rule of the class has the equilibrium (a,b), the closure of the rules of the class with this equilibrium is the set of all such ν that satisfy the conditions, as mixing any of them with a rule of the class gives a rule of the class. This set is weakly compact, so a maximizer exists, and as there are three constraints, an extreme point has at most three atoms. Taking the supremum over the pairs (a,b) that are the equilibrium of some rule of the class gives the claim. This restriction is needed: for $a, b \leq c$, τ_c makes $N \equiv 0$ on $[0,c]$ and satisfies all conditions, so as $c \to 1$ the unrestricted program approaches the first best. □

The supremum is in general not attained within the class: the maximizing mixtures of trapezoid rules have a density that is linear near 0, either strictly decreasing there, as for $v^*(x)$, or flat. When the favorite pools and the underdog reveals fully, the problem has a closed form. Let $\kappa_1 := p_1(1-p_2)$ and $\kappa_2 := p_2(1-p_1)$.

Remark OA.5. Let $v(x)$ be well S-shaped with $v'(x)$ not linear near 0, $p_2 < \min(p_1, \frac{1}{2})$, and let the equilibrium be $(\{p_1\}; \{0,1\})$. Then $p_1 > \frac{1}{2}$ and $u_F \leq \overline{U}$, where

$$\overline{U} := \frac{p_1+p_2}{2} + \begin{cases} \kappa_1 T(p_1) + \kappa_2 T(1-p_1), & p_1 \geq \frac{1}{\sqrt{2}}, \\ \dfrac{\kappa_1(4p_1-1)+\kappa_2}{2(1+2p_1^2)}, & \frac{1}{2} < p_1 < \frac{1}{\sqrt{2}}. \end{cases}$$

For $p_1 \geq \frac{1}{\sqrt{2}}$, $\overline{U}$ is the payoff of $v^*(x)$, and no other rule attains it. For $\frac{1}{2} < p_1 < \frac{1}{\sqrt{2}}$, $\overline{U}$ is approached by rules close to τ_{c^*}, $c^* := \sqrt{\frac{1}{2} - p_1^2}$, if $p_2 < \frac{4p_1^2-1}{6p_1^2-1}$, and not approached if $p_2 > \frac{4p_1^2-1}{6p_1^2-1}$.

So for $\frac{1}{2} < p_1 < \frac{1}{\sqrt{2}}$, the firm blurs comparisons of close signals just enough to make the underdog reveal fully: against a pooling favorite, under $v^*(x)$ he would stop at $\sqrt{2}p_1 < 1$.

Proof. By Steps 1 and 4 of the proof of Proposition 2, $p_1 \geq p^* > \frac{1}{2}$, and the underdog reveals fully if and only if $B(1; p_1) \geq 0$. Normalize v as above; then $u_F = \frac{p_1+p_2}{2} + J(w)$ with $J(w) := \kappa_1 H_w(p_1) + \kappa_2 H_w(1-p_1)$. For $p_1 \geq \frac{1}{\sqrt{2}}$, $H_w \leq T$ by Lemma 6, with equality only for $v^*(x)$, whose equilibrium is $(\{p_1\}; \{0,1\})$ by Remark OA.2(i).

Let $\frac{1}{2} < p_1 < \frac{1}{\sqrt{2}}$, and write $w(t) = \mu([0,t]) = \int \mathbf{1}\{t \geq c\}\, d\mu(c)$ for a measure $\mu \geq 0$ on $[0,1)$. For the step $\mathbf{1}\{t \geq c\}$, let $J(c)$, $B_1(c)$ and $n(c) = \frac{1-c^2}{2}$ be the values of J, $B(1;p_1)$ and $\int_0^1 t w(t) dt$, so that the normalization is $\int n\, d\mu = \frac{1}{2}$. With $J_0 := \kappa_1 T(p_1) + \kappa_2 T(1-p_1)$ and $\kappa := \kappa_1 + \kappa_2$, $H_w(x) = x(1-c)$ for $x \leq c$ and $H_w(x) = T(x) - \frac{c^2}{2}$ for $x \geq c$ for the step w give

$$\begin{aligned}
&J(c) = J_0 - \tfrac{\kappa c^2}{2}, && B_1(c) = p_1^2 + c^2 - \tfrac{1}{2}, && c \leq 1 - p_1,\\
&J(c) = \kappa_1\big(T(p_1) - \tfrac{c^2}{2}\big) + \kappa_2(1-p_1)(1-c), && B_1(c) = \tfrac{(p_1-c)^2}{2}, && 1 - p_1 \leq c \leq p_1,\\
&J(c) = (1-c)\big(\kappa_1 p_1 + \kappa_2(1-p_1)\big), && B_1(c) = 0, && c \geq p_1.
\end{aligned}$$

Let $\eta := \frac{\kappa - 2J_0}{1+2p_1^2}$, positive as $T < \frac{1}{2}$ on $[0,1)$, $\beta := \kappa - 2\eta = \frac{\kappa_1(4p_1-1)+\kappa_2}{1+2p_1^2}$, and $\psi(c) := J(c) + \eta B_1(c) - \beta n(c)$. On $[0, 1-p_1]$, $\psi \equiv 0$. On $[p_1, 1]$, $\psi(c) = (1-c)\big[\kappa_1 p_1 + \kappa_2(1-p_1) - \frac{\beta(1+c)}{2}\big] \leq 0$, as $(1+p_1)\big(\kappa_1(4p_1-1)+\kappa_2\big) - 2(1+2p_1^2)\big(\kappa_1 p_1 + \kappa_2(1-p_1)\big) = \kappa_1(1-p_1)(4p_1^2-1) + \kappa_2(4p_1^3 - 4p_1^2 + 3p_1 - 1) \geq 0$, the cubic being increasing and zero at $\frac{1}{2}$. On $[1-p_1, p_1]$, ψ is quadratic with $\psi(1-p_1) = 0$, right derivative $-\eta < 0$ there and $\psi(p_1) \leq 0$, so $\psi \leq 0$. Hence, as $B(1;p_1) = \int B_1 d\mu \geq 0$,

$$J(w) = \int J\, d\mu \leq \int (J + \eta B_1)\, d\mu = \int (\psi + \beta n)\, d\mu \leq \frac{\beta}{2},$$

which is $u_F \leq \overline{U}$. Equality requires μ to live on $[0, 1-p_1]$ and $B(1;p_1) = 0$, i.e. $\int c^2 d\mu = (\frac{1}{2} - p_1^2)\mu([0,1))$. For such w, $A(p_1) = \int \frac{p_1^2 - c^2}{2} d\mu$ and $B(p_1;1) = \frac{p_1^2}{2}\mu([0,1))$, so the favorite's condition $(1-p_2)A(p_1) \geq p_2 B(p_1;1)$ reads $p_2 \leq \frac{4p_1^2-1}{6p_1^2-1}$ for all maximizers. If it fails, the supremum is below $\overline{U}$: in terms of the probability measure $2n\, d\mu$ on $[0,1]$, as in the proof of Remark OA.4, $\frac{\psi}{n}$ is continuous and negative on $(1-p_1, 1]$, so a maximizing sequence has a weak limit supported on $[0, 1-p_1]$ with $B(1;p_1) = 0$ that satisfies the favorite's condition. If it holds strictly, τ_{c^*} is a maximizer, as $B_1(c^*) = 0$ and $c^* < 1 - p_1$, and shifting a little mass of μ to larger c and adding a small multiple of t to w gives rules of the class in which the equilibrium is still $(\{p_1\}; \{0,1\})$, with payoffs converging to $\overline{U}$. □

Remark OA.6. Let $p_2 < \min(p_1, \frac{1}{2})$. If the favorite pools, every well S-shaped rule with $v'(x)$ not linear near 0 gives the firm less than the noiseless rule if $p_1 \leq 1 - \frac{1}{\sqrt{3}} \approx 0.423$ or the underdog reveals fully.

Proof. If the equilibrium is $(\{p_1\}; \{0, b\})$, then $p_1 < b < \min(2p_1, 1)$, or $b = 1$ and $p_1 > \frac{1}{2}$

(Step 1 of the proof of Proposition 2), so $b - p_1 < p_1$, and as $\Psi(x) = xH_w(x) \leq xT(x)$ by Lemma 6, which increases in x, $u_F \leq \frac{p_1+p_2}{2} + p_1 T(p_1)$. For $p_1 \leq 1 - \frac{1}{\sqrt{3}}$, $T(p_1) \leq \frac{1}{3}$, so this is less than $p_1 + \frac{p_2}{3} = u_F^0$, as $p_2 < p_1$. If the underdog reveals fully, then $u_F \leq \overline{U}$ by Remark OA.5, and $\overline{U} - u_F^0$ is linear in p_2, negative at $p_2 = 0$, as $T < \frac{1}{2}$ and $4p_1 - 1 < 1 + 2p_1^2$, and, at $p_2 = \frac{1}{2}$, equal to the expression for D in the proof of Remark OA.3 if $p_1 \geq \frac{1}{\sqrt{2}}$ and to

$$-\frac{(p_1 - 1)(2p_1 - 1)(3p_1^4 - 7p_1^3 + 3p_1^2 - 3p_1 + 1)}{6p_1^3(1 + 2p_1^2)} < 0$$

if $\frac{1}{2} < p_1 < \frac{1}{\sqrt{2}}$, as the quartic is negative on $[\frac{1}{2}, \frac{1}{\sqrt{2}}]$. □

Numerically, the remark extends to all priors: in the remaining case, the favorite pools, the underdog concedes bad news and $p_1 > 0.423$, an upper bound on the firm's payoff, the linear program of Remark OA.4 without the condition on $v'(x)$, is below u_F^0 on a grid of 705 prior pairs with $p_1 \geq 0.4$. Table OA.1 reports the supremum over well S-shaped rules, computed from the linear programs of Remark OA.4, with the equilibrium enforced by requiring $w - \delta t$ to be non-decreasing for a small $\delta > 0$, and checked against the discretized game. On a grid of priors with step 0.05, it exceeds the payoff of $v^*(x)$ by at most 0.043, and it beats the noiseless rule only on a band of close priors that is slightly wider than that of $v^*(x)$: with $p_1 < 0.263$, 0.332, 0.407, 0.490, 0.583 and 0.675 at $p_2 = 0.25$, 0.3, 0.35, 0.4, 0.45 and 0.48.

Table OA.1: Firm's payoff with unequal priors

(p_1, p_2)	first best	noiseless	$v^*(x)$	sup. well S-shaped
(0.3, 0.2)	0.440	0.367	0.297	0.314
(0.4, 0.2)	0.520	0.467	0.392	0.411
(0.6, 0.3)	0.720	0.696	0.632	0.656
(0.8, 0.3)	0.860	0.858	0.830	0.830
(0.4, 0.35)	0.610	0.517	0.518	0.521
(0.45, 0.35)	0.643	0.567	0.526	0.544
(0.5, 0.45)	0.725	0.650	0.683	0.690
(0.6, 0.4)	0.760	0.728	0.659	0.702
(0.75, 0.45)	0.863	0.856	0.818	0.832

Flat densities. Rules whose density is flat near 0 lie just outside the well S-shaped class. Let $v'(x)$ be even and concave, constant exactly on $[-c, c]$ for some $c \in (0, 1)$, and let $g_0 := v'(0)$.

Remark OA.7. Let $p_2 \leq p_1$. If $p_1 > c$, the equilibrium is unique. If $p_1 \leq c$, the equilibria are all pairs of signals supported on $[0, c]$, and the firm's payoff ranges from

$\frac{p_1+p_2}{2} + g_0(p_1 - p_2)^2$, without disclosure, to $\frac{p_1+p_2}{2} + g_0\big(c(p_1 + p_2) - 2p_1p_2\big)$, when both agents play $\{0, c\}$.

As $g_0 \le \frac{1}{1+c}$, with equality for τ_c (by concavity and $v'(1) \ge 0$, $v'(t) \ge g_0\frac{1-t}{1-c}$ on $[c, 1]$, so $\frac{1}{2} \ge \hat{v}(1) \ge \frac{g_0(1+c)}{2}$), a firm that could pick her preferred equilibrium would get the most, among rules with $c \ge p_1$, from τ_c: $\frac{p_1+p_2}{2} + \frac{c(p_1+p_2)-2p_1p_2}{1+c}$, which rises to the first best as $c \to 1$, as for the linear rule of Proposition 6. The least preferred equilibrium of these rules, however, reveals nothing, so the design problem is not simplified this way. Rules with $c < p_1$ have a unique equilibrium, and numerically, at every prior pair of the grid with step 0.05, the supremum of Remark OA.4 is attained by a mixture of trapezoid rules of this kind.

Proof. Let $w := -v''$ on $(0, 1]$, extended as an odd function: it is non-decreasing, 0 on $(0, c)$ and positive on $(c, 1]$. Against a signal G_j, agent i's value function $W(s) := \mathbb{E}[v(s - s_j)]$ has $W''(s) = \mathbb{E}[w(s_j - s)]$, non-increasing in s, and $N_W(y) := yW'(y) - W(y) + W(0) = \int_0^y sW''(s)ds$.

Let $p_1 > c$. For $s \in [0, c]$, $s_1 - s \ge -c$, so $W_2'' \ge 0$ on $[0, c]$, and $W_2'' > 0$ near 0, as $\mathbb{P}(s_1 > c) > 0$. So $N_{W_2} > 0$ on $(0, c]$; it rises while $W_2'' > 0$, is constant while $W_2'' = 0$ and falls strictly once $W_2'' < 0$, so it has at most one zero b on $(c, 1]$, and agent 2's best reply is unique: $\{0, b\}$, $\{0, 1\}$ if there is no zero, or $\{p_2\}$ if $p_2 \ge b$. Each of these has mass above c. As the equilibria form a product set, agent 2's equilibrium signal is unique and has mass above c, and the same argument makes agent 1's best reply to it unique.

Let $p_1 \le c$. Since $v' \le g_0$ with equality exactly on $[-c, c]$, $\hat{v}(x) \le g_0 x$ for $x \ge 0$, with equality if and only if $x \le c$, and $\hat{v}(x) \ge g_0 x$ for $x \le 0$, with equality if and only if $x \ge -c$. If G_j is supported on $[0, c]$, then $W_i(s) \le L(s) := \frac{1}{2} + g_0(s - p_j)$ on $[0, 1]$, with equality on $[0, c]$, so every signal of agent i supported on $[0, c]$ is a best reply and wins with probability $v(p_i - p_j)$. Hence these pairs are equilibria, with value $v(p_1 - p_2)$ for agent 1. Conversely, let G_2 put mass above c, and let $L(s) := \frac{1}{2} + g_0(s - p_2)$. Then $D_0 := W_1(0) - L(0) > 0$ and $W_1(p_1) \ge L(p_1) = v(p_1 - p_2)$, with equality only if $s_2 \le p_1 + c$ almost surely; in that case $W_1 = L$ on $[p_1, c]$ and $L(c + \delta) - W_1(c + \delta) \le \int_c^{c+\delta}(g_0 - v') = o(\delta)$. So pooling or a signal $\{0, c'\}$ with c' at or slightly above c wins with probability above $v(p_1 - p_2)$, and G_2 is not an equilibrium signal; the same holds for agent 1. Finally, on $[-c, c]$, $\Psi(x) = g_0x^2$, so $u_F = \frac{p_1+p_2}{2} + g_0\big(\mathrm{Var}[s_1] + \mathrm{Var}[s_2] + (p_1 - p_2)^2\big)$ with $0 \le \mathrm{Var}[s_i] \le p_i(c - p_i)$. □

OA.3. General Priors

OA.3.1 Full revelation and the top atom

Lemma OA.5. *Let $v(x)$ be well S-shaped and F a prior on $[0,1]$ with $\underline{\theta} < \overline{\theta}$, and let $\alpha := F(\overline{\theta}) - F(\overline{\theta}-)$.*

(i) If $\overline{\theta}$ is isolated in $\operatorname{supp}(F)$, let $t := \max\big(\operatorname{supp}(F) \cap [0, \overline{\theta})\big)$. Then $\alpha \geq 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(\overline{\theta}-t)}$, and if $\operatorname{supp}(F) = \{t, \overline{\theta}\}$, this condition is also sufficient.

(ii) Otherwise, F is a best reply to F if and only if $\alpha|\tilde{v}''_+(0)| \geq \int_{[0,\overline{\theta})} |\tilde{v}''_-(\overline{\theta} - y)|\, dF(y)$; this requires $\tilde{v}''_+(0) \neq 0$.

Proof. Necessity in (i) and (ii) is shown in Cases 1 and 2 of the proof of Lemma 11, whose notation w, g and L I use. Sufficiency in (i): If $\operatorname{supp}(F) = \{t, \overline{\theta}\}$, every feasible signal is supported on $[t, \overline{\theta}]$, and the condition is equivalent, as in the proof of Proposition 1, to the secant ℓ of w from t to $\overline{\theta}$ being weakly flatter than $w'(\overline{\theta})$. Then $D := \ell - w$ has $D(t) = D(\overline{\theta}) = 0$ and a convex derivative with $D'(\overline{\theta}) \leq 0$, so D' is positive and then non-positive on $[t, \overline{\theta}]$, $D \geq 0$ there, and $\int w\, dG' \leq \int \ell\, dG' = \int \ell\, dF = \int w\, dF$ for every feasible G'.

Sufficiency in (ii): Conversely, if the condition holds, then $g'_-(\overline{\theta}) \geq 0$, so the concave g is non-decreasing on $[0, \overline{\theta}]$, and F is a best reply to F by Step 2 of the proof of Lemma 10. □

For $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$, $\tilde{v}(x) = -x|x|$, so both bounds read $\alpha \geq \frac{1}{2}$, and by Appendix C.4 full revelation is an equilibrium if and only if $\alpha \geq \frac{1}{2}$. For rules with $\tilde{v}''_+(0) = 0$, such as those generated by normal or Gumbel shocks, (ii) excludes full revelation whenever types accumulate at $\overline{\theta}$. For the binary prior, (i) is Proposition 1: $t = 0$, $\overline{\theta} = 1$ and $\alpha = p$.

OA.3.2 Equality in Proposition 9

Proposition OA.5. *In Proposition 9, the utility bound holds with equality if and only if $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$, F is degenerate, or $v(1) = 1$ and F puts mass $p \geq \overline{p}$ on 1 and $1 - p$ on 0.*

Proof. Steps 1–3 refer to the proof of the bounds in Proposition 9 in Appendix C.6. Equality requires $U(q) = U(\frac{1}{2})$, i.e. $G = G_{1/2}$ by Steps 1 and 3, and equality in Step 2. If $v \neq v^*$, the latter means that $|s_1 - s_2| \in \{0, 1\}$ almost surely under $G_{1/2} \times G_{1/2}$, and $v(1) = 1$ if the gap 1 has positive probability. So $\operatorname{supp}(G_{1/2})$ is a single point or

$\{0,1\}$. In the first case, the revealed types and the pooled mean $m(\frac{1}{2})$ coincide, so F is degenerate. In the second, $m(\frac{1}{2}) = 1$, as it is the largest point of the support, so $x_u = 1$ for $u > \frac{1}{2}$ and $x_u \in \{0,1\}$ for $u < \frac{1}{2}$: F puts mass $p \geq \frac{1}{2}$ on 1 and $1-p$ on 0, and $G = F$, which by Proposition 1 requires $p \geq \overline{p}$; moreover $v(1) = 1$. Conversely, equality holds in these cases: trivially if F is degenerate, and for the binary prior with $p \geq \overline{p}$ and $v(1) = 1$, both $v(x)$ and $v^*(x)$ induce full revelation, as $\overline{p} \geq \frac{1}{2}$ is the threshold of $v(x)$ and $\frac{1}{2}$ that of $v^*(x)$, and $u_F = p + 2p(1-p)\big(v(1) - \frac{1}{2}\big) = 2p - p^2$ under both. □

OA.3.3 A single pass mark for the best fully revealing rule

Lemma OA.6. *Let F be any prior. L is generated by a single random pass mark C, drawn independently of the signals, under which the higher signal is selected if it is at least C and a coin is tossed otherwise, if and only if $\sqrt{I_F}$ is concave on $[\underline{\theta}, \overline{\theta}]$; the pass mark then has the CDF $\mathbb{P}(C \leq z) = \frac{I_F(z)}{(\overline{\theta}-p)\,F(z)^2}$. Every concave prior satisfies this condition.*

Proof. Under a pass mark with CDF K, $H(x,y) = \frac{1}{2} + K(\max(x,y))\big(\mathbb{1}\{x > y\} - \frac{1}{2}\big)$, which is nondecreasing in x, and $V_F(x) = \frac{1}{2} + \frac{1}{2}F(x)K(x) - \frac{1}{2}\int_x^{\overline{\theta}} K\,dF$. So $V_F' = Kf + \frac{1}{2}FK'$, i.e., $(F^2K)' = 2F\,V_F'$, and $V_F = L$ if and only if $F^2K = I_F/(\overline{\theta} - p)$, using $I_F(\overline{\theta}) = \overline{\theta} - p$ for $K(\overline{\theta}) = 1$. This K is a CDF if and only if $I_F/F^2 = I_F/(I_F')^2$ is nondecreasing, i.e., $(I_F')^2 \geq 2I_F I_F''$, i.e., $\sqrt{I_F}$ is concave. If F is concave, it lies below its tangent at z, which vanishes at or below $\underline{\theta}$, so $I_F(z) \leq \frac{F(z)^2}{2f(z)}$, which is the same inequality. □

So for concave priors the best fully revealing rule is a lottery among agents who *fail* a random pass mark. A lottery among agents who *clear* a bar instead makes V_F flat at the top, so V_F is not convex and agents gain by pooling their best types.

OA.3.4 Robust full revelation

The rule H^* that draws a bar uniformly between the lower signal and 1, selects the higher signal if it clears the bar, and tosses a coin otherwise makes full revelation the unique equilibrium for every prior. Among the rules under which full revelation is an equilibrium for every prior, it selects the higher signal with the highest probability at every pair of signals, so it is the best such rule for every prior (Proposition OA.6).

Proposition OA.6. *If a general selection rule H has $H(\cdot, y)$ convex for every y, full revelation is an equilibrium for every prior on $[0,1]$; conversely, if H is continuous and full revelation is*

an equilibrium for every prior, each $H(\cdot, y)$ is convex. Among such rules,

$$H^*(x, y) = \frac{1}{2} + \frac{x - y}{2\big(1 - \min(x, y)\big)}$$

selects the higher signal with the highest probability at every pair of signals, so it maximizes the firm's utility under full revelation for every prior simultaneously. For every prior on $[0, 1]$, full revelation is the unique equilibrium under H^*.

Proof. Sufficiency: convexity of each $H(\cdot, y)$ makes V_F convex for every F, so full revelation is a best reply to itself by Jensen's inequality. Necessity: for atomless full-support priors $F_n \to \delta_y$, full revelation is a best reply to itself, so pooling the types in any interval does not pay, and V_{F_n} is convex by Lemma 12; as H is continuous, V_{F_n} converges pointwise to $H(\cdot, y)$, which is therefore convex. Optimality: for $x > y$, $H(\cdot, y)$ is convex on $[y, 1]$ with $H(y, y) = \frac{1}{2}$ and $H(1, y) \leq 1$, so it lies below the chord, $H(x, y) \leq \frac{1}{2} + \frac{x-y}{2(1-y)} = H^*(x, y)$. H^* is admissible: for $x < y$, $H^*(x, y) = \frac{1-y}{2(1-x)}$, which is convex in x and has the same slope $\frac{1}{2(1-y)}$ at $x = y$ as the linear branch, and $H^*(x, y) + H^*(y, x) = 1$. Since $u_F = \mathbb{E}[m + (M - m)H(M, m)]$ with $M = \max(\theta_1, \theta_2)$ and $m = \min(\theta_1, \theta_2)$, pointwise dominance gives the ranking. Uniqueness: H^* is continuous except at $(1, 1)$, so V_F is continuous on $[\underline{\theta}, 1)$, and $V_F''(x) = \int_{(x,1]} \frac{1-y}{(1-x)^3} dF(y) > 0$ on $[\underline{\theta}, t)$, where $t := \sup(\mathrm{supp}(F) \cap [0, 1))$; on $[t, 1)$, which contains no type, V_F is affine, and if F has an atom at 1, V_F jumps up there, as $H^*(x, 1) = 0$ for $x < 1$ and $H^*(1, 1) = \frac{1}{2}$. A mean-preserving contraction $G \neq F$ pools some types $a < b$; if $a < t$, strict convexity on $[a, \min(b, t))$ makes the pooling strictly costly, and otherwise $a = t$ and $b = 1$, and the jump at 1 does. So $\int V_F \, dG < \int V_F \, dF$, F is the unique best reply to F, and by interchangeability every equilibrium (G_1, G_2) has (G_1, F) as an equilibrium, so $G_1 = F$, and likewise $G_2 = F$. □

The bound of Proposition 11(i) is also approached by rules with full revelation as the unique equilibrium. Under L, the agent is indifferent among all feasible signals; mixing the rule with H^* of Proposition OA.6, whose V_F is strictly convex below the top type (see its proof), makes full revelation the unique equilibrium at a payoff arbitrarily close to the bound.

OA.3.5 Single-peaked and U-shaped priors

Proposition OA.7. *Let F have a density that is positive on $(\underline{\theta}, \overline{\theta})$, nondecreasing below some $m \in [\underline{\theta}, \overline{\theta}]$ and nonincreasing above it. Among general selection rules nondecreasing in the*

own signal, with upper semicontinuous V_F, under which full revelation is an equilibrium, a rule maximizes the firm's utility if and only if, F-almost surely,

(i) $V_F = L$, *if* $p \leq \frac{\underline{\theta}+\overline{\theta}}{2}$;

(ii) $V_F = F$, *if* $p > \frac{\underline{\theta}+\overline{\theta}}{2}$ *and* F *is convex;*

(iii) otherwise, $V_F(\theta) = F(\theta)$ *for* $\theta \leq c^*$ *and* $V_F(\theta) = F(c^*) + \big(1 - F(c^*)\big)\frac{\theta - c^*}{\overline{\theta} - c^*}$ *for* $\theta \geq c^*$, *where* c^* *is the smallest solution in* $(\underline{\theta}, m]$ *of* $\mathbb{E}_F[\theta \mid \theta \geq c] = \frac{c+\overline{\theta}}{2}$, *and* $c^* < m$.

Let F have a density f that is positive on $(\underline{\theta}, \overline{\theta})$. For a rule nondecreasing in the own signal, V_F is nondecreasing. As $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$, full revelation is an equilibrium under a rule with upper semicontinuous V_F if and only if V_F is convex on $[\underline{\theta}, \overline{\theta}]$, by Jensen's inequality and Lemma 13. By Lemmas 15 and 17(i), a nondecreasing V equals V_F for some rule nondecreasing in the own signal if and only if $V(\theta) \leq_{cx} U[0,1]$, and by Lemma 14, $u_F = 2\int \theta\, V_F(\theta)\, dF(\theta)$. As F is atomless, the value of V_F at $\overline{\theta}$ does not matter. So Proposition OA.7 concerns the problem (FRP) below, which I first solve for every such F and then for single-peaked F. For a bounded measurable V on $[\underline{\theta}, \overline{\theta}]$ let

$$T_V(\theta) := \int_{[\theta,\overline{\theta}]} V\, dF, \qquad M(\theta) := \frac{1 - F(\theta)^2}{2} = T_F(\theta), \qquad \theta \in [\underline{\theta}, \overline{\theta}].$$

Let C be the set of convex nondecreasing $V : [\underline{\theta}, \overline{\theta}] \to [0,1]$; such a V is continuous on $[\underline{\theta}, \overline{\theta})$, and I always take $V(\overline{\theta}) := V(\overline{\theta}-)$, which changes nothing below. For $V \in C$ let $V'(\underline{\theta}^+)$ be its right derivative at $\underline{\theta}$ and ν_V its second-derivative measure on $(\underline{\theta}, \overline{\theta})$ (the Lebesgue–Stieltjes measure of the right derivative), so that

$$V(\theta) = V(\underline{\theta}) + V'(\underline{\theta}^+)(\theta - \underline{\theta}) + \int_{(\underline{\theta},\overline{\theta})} (\theta - a)^+\, d\nu_V(a), \qquad \theta \in [\underline{\theta}, \overline{\theta}]. \tag{OA.1}$$

The feasible set of the fully revealing problem is

$$\mathcal{V} := \Big\{ V \in C : \int V\, dF = \tfrac{1}{2},\ \ T_V \leq M \text{ on } [\underline{\theta}, \overline{\theta}] \Big\},$$

and the problem is

$$\text{(FRP)} \qquad \max_{V \in \mathcal{V}} \int \theta\, V(\theta)\, dF(\theta).$$

For $V \in \mathcal{V}$ let $B(V) := \{\theta \in (\underline{\theta}, \overline{\theta}) : T_V(\theta) = M(\theta)\}$ be the set where the implementability constraint binds. Integrating by parts,

$$\int \theta V \, dF = \frac{\overline{\theta}}{2} + \int_{\underline{\theta}}^{\overline{\theta}} T_V(\theta) \, d\theta, \tag{OA.2}$$

so the firm wants T_V as close to its cap $M = T_F$ as possible; without convexity the cap is attained by $V = F$, and convexity is what forces T_V below M.

Lemma OA.7. *(i) $\mathcal{V}$ is nonempty and convex, compact for pointwise convergence, and the maximum in (FRP) is attained.*

(ii) If $V \in \mathcal{V}$ and $\theta \in B(V)$, then $V(\theta) = F(\theta)$, $\int_x^\theta (V - F)\, dF \le 0$ for $x < \theta$ and $\int_\theta^x (V - F)\, dF \ge 0$ for $x > \theta$.

(iii) For nondecreasing $V : [\underline{\theta}, \overline{\theta}] \to [0,1]$ with $\int V \, dF = \frac{1}{2}$, $V(\theta) \le_{cx} U[0,1]$ if and only if $T_V \le M$ on $[\underline{\theta}, \overline{\theta}]$.

Proof. (i) $V \equiv \frac{1}{2}$ is feasible, as $T_{1/2} - M = -\frac{1}{2}F(1-F) \le 0$. By Helly's theorem every sequence in $\mathcal{C}$ has a subsequence converging pointwise on $[\underline{\theta}, \overline{\theta})$ to some $V \in \mathcal{C}$; F is atomless, so $\int V_n \, dF \to \int V \, dF$ and $T_{V_n} \to T_V$ pointwise by dominated convergence, and so does the objective. Hence $\mathcal{V}$ is compact and the objective is continuous on it. (ii) For $x < \theta$, $T_V(x) - M(x) = T_V(\theta) - M(\theta) + \int_x^\theta (V - F)\, dF = \int_x^\theta (V - F)\, dF \le 0$; dividing by $F(\theta) - F(x) > 0$ and letting $x \uparrow \theta$ gives $V(\theta) \le F(\theta)$ by continuity. For $x > \theta$, $T_V(x) - M(x) = -\int_\theta^x (V - F)\, dF \le 0$ gives $V(\theta) \ge F(\theta)$. (iii) For random variables with the same mean and values in $[0,1]$, $V(\theta) \le_{cx} U[0,1]$ if and only if $\mathbb{E}_F[(V-t)^+] \le \mathbb{E}[(U-t)^+] = \frac{(1-t)^2}{2}$ for all $t \in [0,1]$ (for $t \notin [0,1]$ both sides agree). As V is nondecreasing, its upper level sets are intervals $[\theta, \overline{\theta}]$, so $\mathbb{E}_F[(V-t)^+] = \sup_\theta \int_{[\theta,\overline{\theta}]} (V - t)\, dF = \sup_\theta \big(T_V(\theta) - t(1 - F(\theta))\big)$. So the condition is $T_V(\theta) \le \inf_{t \in [0,1]} \left[\frac{(1-t)^2}{2} + t(1 - F(\theta))\right] = M(\theta)$ for all θ, the infimum being attained at $t = F(\theta)$. □

By Lemma OA.7(iii), the constraint $T_V \le M$ is Border's condition $V(\theta) \le_{cx} U[0,1]$.

A dual characterization. A *multiplier* is a triple (K, λ, D): a nondecreasing $K : (\underline{\theta}, \overline{\theta}) \to \mathbb{R}$ with $\int |K| \, dF < \infty$, a constant $\lambda \in \mathbb{R}$, and a constant $D \ge 0$. Let μ be the Lebesgue–Stieltjes measure of K on $(\underline{\theta}, \overline{\theta})$ (a Radon measure, possibly infinite near the endpoints),

and define

$$\psi(s) := s - K(s) - \lambda, \qquad \Psi(\theta) := \int_{\theta}^{\overline{\theta}} \psi \, dF, \qquad \Psi_1(a) := \int_{a}^{\overline{\theta}} \Psi(\theta) \, d\theta, \qquad E(a) := \Psi_1(a) - D(\overline{\theta} - a).$$

Then $\psi \in L^1(F)$, Ψ is absolutely continuous with $\Psi' = -\psi f$ a.e., $E \in C^1[\underline{\theta}, \overline{\theta}]$ with $E' = D - \Psi$, $E'' = \psi f$ a.e., and $E(\overline{\theta}) = 0$, $E'(\overline{\theta}) = D$. Let $\mathcal{K} := \{a \in [\underline{\theta}, \overline{\theta}] : E(a) = 0\}$, which is a closed set containing $\overline{\theta}$.

The characterization rests on two identities.

Lemma OA.8. *Let (K, λ, D) be a multiplier and W nondecreasing with values in $[0,1]$ and $\int W \, dF = \frac{1}{2}$. Then*

$$\int \theta W \, dF = \int W \psi \, dF + \int_{(\underline{\theta}, \overline{\theta})} \big(T_W - M\big) \, d\mu + \frac{\lambda}{2} + \int F \, K \, dF, \tag{OA.3}$$

all integrals being absolutely convergent.

Proof. As $\theta = \psi(\theta) + K(\theta) + \lambda$ and $\int W \, dF = \int F \, dF = \frac{1}{2}$, $\int \theta W \, dF = \int W \psi \, dF + \int (W - F) K \, dF + \int F K \, dF + \frac{\lambda}{2}$. Fix $s_0 \in (\underline{\theta}, \overline{\theta})$. For $s > s_0$, $K(s) = K(s_0) + \mu((s_0, s])$, and for $s \le s_0$, $K(s) = K(s_0) - \mu((s, s_0])$. As $\int (W - F) \, dF = 0$, the constant $K(s_0)$ drops out, and

$$\int (W - F) K \, dF = \int_{s > s_0} (W - F)(s) \, \mu((s_0, s]) \, dF(s) - \int_{s \le s_0} (W - F)(s) \, \mu((s, s_0]) \, dF(s).$$

Both double integrals converge absolutely: $|W - F| \le 1$, and $\int_{s > s_0} \mu((s_0, s]) \, dF(s) = \int_{(s_0, \overline{\theta})} (1 - F) \, d\mu \le \int_{s_0}^{\overline{\theta}} |K - K(s_0)| \, dF < \infty$, and likewise below s_0. By Fubini the first term equals $\int_{(s_0, \overline{\theta})} \big[\int_{[\theta, \overline{\theta}]} (W - F) \, dF \big] d\mu(\theta) = \int_{(s_0, \overline{\theta})} (T_W - M) \, d\mu$, and the second equals $-\int_{(\underline{\theta}, s_0]} \big[\int_{[\underline{\theta}, \theta)} (W - F) \, dF \big] d\mu(\theta) = \int_{(\underline{\theta}, s_0]} (T_W - M) \, d\mu$, as $\int_{[\underline{\theta}, \theta)} (W - F) \, dF = -(T_W - M)(\theta)$. □

Lemma OA.9. *Let (K, λ, D) be a multiplier with $E \le 0$ on $[\underline{\theta}, \overline{\theta}]$ and $\Psi(\underline{\theta}) \le D$. For every $W \in C$,*

$$\int W \psi \, dF = W(\underline{\theta}) \, \Psi(\underline{\theta}) + W'(\underline{\theta}^+) \, \Psi_1(\underline{\theta}) + \int_{(\underline{\theta}, \overline{\theta})} \Psi_1(a) \, d\nu_W(a) \;\le\; D \, W(\overline{\theta}) \;\le\; D, \tag{OA.4}$$

with equality throughout if and only if $\nu_W(\{E < 0\}) = 0$, $E(\underline{\theta}) = 0$ if $W'(\underline{\theta}^+) > 0$, $\Psi(\underline{\theta}) = D$ if $W(\underline{\theta}) > 0$, and $W(\overline{\theta}) = 1$ if $D > 0$.

Proof. Integrating by parts, for $a \in [\underline{\theta}, \overline{\theta})$, $\int (\theta - a)^+ \psi \, dF = \int_a^{\overline{\theta}} (\theta - a)(-\Psi'(\theta)) \, d\theta = \Psi_1(a)$, and $\int \psi \, dF = \Psi(\underline{\theta})$. Insert (OA.1) into $\int W\psi \, dF$; Fubini applies because $\int (\theta - a)^+ dv_W(a) = W(\theta) - W(\underline{\theta}) - W'(\underline{\theta}^+)(\theta - \underline{\theta}) \leq 1$ and $\psi \in L^1(F)$. This gives the identity in (OA.4). By (i), $\Psi(\underline{\theta}) \leq D$ and $\Psi_1(a) = E(a) + D(\overline{\theta} - a) \leq D(\overline{\theta} - a)$ for all $a \in [\underline{\theta}, \overline{\theta})$; as $W(\underline{\theta})$, $W'(\underline{\theta}^+)$ and v_W are nonnegative and $D \geq 0$,

$$\int W\psi \, dF \leq D\Big[W(\underline{\theta}) + W'(\underline{\theta}^+)(\overline{\theta} - \underline{\theta}) + \int (\overline{\theta} - a) \, dv_W(a)\Big] = D\, W(\overline{\theta}) \leq D,$$

by (OA.1) at $\theta = \overline{\theta}$. Equality in the first inequality holds if and only if each weight is positive only where its coefficient is maximal: $W(\underline{\theta}) > 0 \Rightarrow \Psi(\underline{\theta}) = D$; $W'(\underline{\theta}^+) > 0 \Rightarrow \Psi_1(\underline{\theta}) = D(\overline{\theta} - \underline{\theta})$, i.e. $E(\underline{\theta}) = 0$; and $v_W(\{E < 0\}) = 0$. Equality in the second holds if and only if $D = 0$ or $W(\overline{\theta}) = 1$. □

Lemma OA.10. *Let F have a density that is positive on $(\underline{\theta}, \overline{\theta})$, and let (K, λ, D) be a multiplier with $E \leq 0$ on $[\underline{\theta}, \overline{\theta}]$ and $\Psi(\underline{\theta}) \leq D$. Every $W \in \mathcal{V}$ satisfies*

$$\int \theta W \, dF \leq D + \frac{\lambda}{2} + \int F\, K \, dF, \tag{OA.5}$$

with equality if and only if

(i) μ is carried by $B(W)$, i.e. $\mu\big((\underline{\theta}, \overline{\theta}) \setminus B(W)\big) = 0$;

(ii) $v_W(\{E < 0\}) = 0$; $E(\underline{\theta}) = 0$ if $W'(\underline{\theta}^+) > 0$; $\Psi(\underline{\theta}) = D$ if $W(\underline{\theta}) > 0$; and $W(\overline{\theta}) = 1$ if $D > 0$.

In particular, a $W \in \mathcal{V}$ that satisfies (i) and (ii) solves (FRP), and then every solution of (FRP) satisfies (i) and (ii).

Proof of Lemma OA.10. Let $W \in \mathcal{V}$. By Lemmas OA.8 and OA.9, $\int \theta W \, dF = \int W\psi \, dF + \int (T_W - M) \, d\mu + \frac{\lambda}{2} + \int FK \, dF$, where $\int W\psi \, dF \leq D$ and, as $T_W \leq M$ and $\mu \geq 0$, $\int (T_W - M) \, d\mu \leq 0$. So (OA.5) holds, with equality if and only if $\int (T_W - M) \, d\mu = 0$, which is (i), and $\int W\psi \, dF = D$, which is (ii) by Lemma OA.9. A $W \in \mathcal{V}$ that satisfies (i) and (ii) attains the bound, so it solves (FRP), and then so does every solution. □

Call F *single-peaked with mode $m \in [\underline{\theta}, \overline{\theta}]$* if f is positive on $(\underline{\theta}, \overline{\theta})$, nondecreasing on $(\underline{\theta}, m]$ and nonincreasing on $[m, \overline{\theta})$; so F is convex on $[\underline{\theta}, m]$ and concave on $[m, \overline{\theta}]$. Concave priors ($m = \underline{\theta}$) and convex priors ($m = \overline{\theta}$) are included; L and $\bar{U}$ are as in Subsection 5.2.

For $c \in [\underline{\theta}, \overline{\theta})$ write $F_c := F(\cdot \mid \theta \geq c)$ for the prior truncated from below at c and $p_c := \mathbb{E}_F[\theta \mid \theta \geq c]$.

Proof of Proposition OA.7. Let F be single-peaked with mode m. By the reduction above, the proposition says (FRP) has a unique solution V^*, the function in (i), (ii) or (iii).

(i) By Lemma 19(i), $L \in \mathcal{V}$; by Proposition 11, as only $V_F = L$ attains its bound, $V^* = L$.

(ii) *Convex F.* If F is convex, then $F \in \mathcal{V}$ ($T_F = M$), and by (OA.2) every $W \in \mathcal{V}$ has $\int \theta W \, dF \leq \frac{\theta}{2} + \int_{\underline{\theta}}^{\overline{\theta}} M \, d\theta = \int \theta F \, dF$, with equality only if the continuous functions T_W and M coincide, i.e. only if $W = F$ F-almost everywhere; so $V^* = F$.

(iii) From now on $p > \frac{\underline{\theta}+\overline{\theta}}{2}$ and F is not convex; then $m < \overline{\theta}$, as otherwise f is nondecreasing on $(\underline{\theta}, \overline{\theta})$.

Existence of c^*, and $c^* < m$. Let $\phi(c) := p_c - \frac{c+\overline{\theta}}{2}$ on $[\underline{\theta}, \overline{\theta})$, continuous, with $\phi(\underline{\theta}) = p - \frac{\underline{\theta}+\overline{\theta}}{2} > 0$. For $c \geq m$ the density of F_c is nonincreasing on $[c, \overline{\theta}]$, so $p_c \leq \frac{c+\overline{\theta}}{2}$ and $\phi(m) \leq 0$. Hence the zero set of ϕ in $(\underline{\theta}, m]$ is nonempty and closed in $[\underline{\theta}, m]$ and does not contain $\underline{\theta}$, so it has a smallest element $c^* \in (\underline{\theta}, m]$. If $\phi(m) < 0$, then $c^* < m$ by continuity. Suppose $\phi(m) = 0$ and let $\omega := \frac{m+\overline{\theta}}{2}$, $h := \frac{\overline{\theta}-m}{2}$. Then $0 = (1 - F(m))\phi(m) = \int_0^h t\,[f(\omega + t) - f(\omega - t)]\, dt$, and the integrand is ≤ 0 as f is nonincreasing on $[m, \overline{\theta})$; so $f(\omega + t) = f(\omega - t)$ for a.e. $t \in (0, h)$, and by monotonicity f is a.e. equal to a constant k on $(m, \overline{\theta})$. If $f(m^-) \leq k$, then f is nondecreasing on $(\underline{\theta}, \overline{\theta})$ and F is convex, which is excluded; so $f(m^-) > k$. Let $N(c) := (1 - F(c))\phi(c) = \int_c^{\overline{\theta}} \left(\theta - \frac{c+\overline{\theta}}{2}\right) dF(\theta)$. It is absolutely continuous with $N'(c) = \frac{\overline{\theta}-c}{2} f(c) - \frac{1-F(c)}{2}$ a.e., and as $c \uparrow m$, $f(c) \to f(m^-)$ and $1 - F(c) \to k(\overline{\theta} - m)$, so $N'(c) \to \frac{\overline{\theta}-m}{2}\left(f(m^-) - k\right) > 0$. Hence $N' > 0$ a.e. on some $(m - \delta, m)$, and $N(c) = N(m) - \int_c^m N' < 0$ there; so $\phi < 0$ on $(m - \delta, m)$, ϕ has a zero in $(\underline{\theta}, m - \delta)$, and $c^* < m$.

Feasibility. Fix a zero $c \in (\underline{\theta}, m)$ of ϕ (for instance c^*), let $q := F(c)$ and let V be the function in (iii) with c in place of c^*. On $[c, \overline{\theta}]$, $V = q + (1 - q)U_c$ with $U_c(\theta) := \frac{\theta-c}{\overline{\theta}-c}$ the CDF of $U[c, \overline{\theta}]$, and $F = q + (1 - q)F_c$, so for $\theta \geq c$

$$T_V(\theta) - M(\theta) = \int_\theta^{\overline{\theta}} (V - F)\, dF = (1 - q)^2 \int_\theta^{\overline{\theta}} (U_c - F_c)\, dF_c.$$

Now F_c is single-peaked on $[c, \overline{\theta}]$ (its density is $f/(1 - q)$ and $c \leq m$) with mean $p_c = \frac{c+\overline{\theta}}{2}$, the midpoint of its support, and U_c is the function L of Lemma 19(i) for

the prior F_c (indeed $\frac{1}{2} + \frac{\theta - p_c}{2(\overline{\theta} - p_c)} = \frac{\theta - c}{\overline{\theta} - c}$). By Lemma 19(i) and Lemma OA.7(iii) applied to F_c, $\int_\theta^{\overline{\theta}} (U_c - F_c)\, dF_c \le 0$ for all $\theta \ge c$, with equality at $\theta = c$. So $T_V \le M$ on $[c, \overline{\theta}]$ and $T_V(c) = M(c)$; for $\theta \le c$, $T_V(\theta) - M(\theta) = T_V(c) - M(c) + \int_\theta^c (F - F)\, dF = 0$. In particular $\int V\, dF = T_V(\underline{\theta}) = \frac{1}{2}$, and $B(V) \supseteq [\underline{\theta}, c]$. V is nondecreasing with values in $[0, 1]$, convex on $[\underline{\theta}, c]$ (as F is) and affine on $[c, \overline{\theta}]$; at c the slope rises: the chord slope $s_c := \frac{1-q}{\overline{\theta} - c}$ satisfies $s_c \ge f(c^+)$, since otherwise, F being convex on $[c, m]$ with right derivative $f(c^+)$ at c, $V(\theta) - F(\theta) \le (s_c - f(c^+))(\theta - c) < 0$ for $\theta \in (c, m]$, and then $T_V(\theta) - M(\theta) = -\int_c^\theta (V - F)\, dF > 0$ for $\theta \in (c, m]$; and $f(c^+) \ge f(c^-)$ because $c < m$ and f is nondecreasing on $(\underline{\theta}, m]$. So $s_c \ge f(c^-)$, V is convex, and $V \in \mathcal{V}$.

A multiplier. Let

$$\kappa := \frac{\int_c^{\overline{\theta}} (\overline{\theta} - s)\, s\, dF(s)}{\int_c^{\overline{\theta}} (\overline{\theta} - s)\, dF(s)} \in [c, \overline{\theta}],$$

and let $K(s) := s - \underline{\theta}$ for $s \in (\underline{\theta}, c]$ and $K(s) := \kappa - \underline{\theta}$ for $s \in (c, \overline{\theta})$; $\lambda := \underline{\theta}$. So μ is Lebesgue measure on $(\underline{\theta}, c)$ plus an atom of mass $\kappa - c$ at c, carried by $[\underline{\theta}, c] \subseteq B(V)$: this is (i) of Lemma OA.10. Then $\psi = 0$ on $(\underline{\theta}, c)$ and $\psi(s) = s - \kappa$ on $(c, \overline{\theta})$. Set $D := \Psi(c) = \int_c^{\overline{\theta}} (s - \kappa)\, dF(s)$. Since $\int_c^{\overline{\theta}} (\overline{\theta} - s)(s - \kappa)\, dF = 0$ by the choice of κ, the function $(s - \kappa)(\overline{\theta} - s) f(s)$ is negative on (c, κ) and positive on $(\kappa, \overline{\theta})$ and integrates to zero, while $1/(\overline{\theta} - s)$ is increasing; hence $D = \int_c^{\overline{\theta}} (s - \kappa)(\overline{\theta} - s) \frac{1}{\overline{\theta} - s}\, dF \ge \frac{1}{\overline{\theta} - \kappa} \int_c^{\overline{\theta}} (s - \kappa)(\overline{\theta} - s)\, dF = 0$, with strict inequality because $f > 0$. So $D > 0$.

The hypothesis and condition (ii) of Lemma OA.10. On $[\underline{\theta}, c]$, $\Psi \equiv \Psi(c) = D$, so $\Psi_1(a) = D(c - a) + \Psi_1(c)$ and $\Psi_1(c) = \int_c^{\overline{\theta}} (s - c)(s - \kappa)\, dF = \int_c^{\overline{\theta}} (\overline{\theta} - c)(s - \kappa)\, dF - \int_c^{\overline{\theta}} (\overline{\theta} - s)(s - \kappa)\, dF = D(\overline{\theta} - c)$; hence $E \equiv 0$ on $[\underline{\theta}, c]$ and $\Psi(\underline{\theta}) = D$. On $[c, \overline{\theta}]$, $E(c) = 0$, $E'(c) = D - \Psi(c) = 0$, $E(\overline{\theta}) = 0$, and $E'' = (s - \kappa) f$ is ≤ 0 on (c, κ) and ≥ 0 on $(\kappa, \overline{\theta})$: E is concave on $[c, \kappa]$ with $E(c) = E'(c) = 0$, so $E \le 0$ there, and convex on $[\kappa, \overline{\theta}]$ with $E(\kappa) \le 0 = E(\overline{\theta})$, so $E \le 0$ there. This is the hypothesis. For (ii): ν_V is carried by $[\underline{\theta}, c] \subseteq \mathcal{K}$ (it is ν_F on $(\underline{\theta}, c)$ plus the atom $s_c - f(c^-)$ at c); $E(\underline{\theta}) = 0$; $V(\underline{\theta}) = 0$; and $V(\overline{\theta}) = 1$. By Lemma OA.10, V solves (FRP), and every solution W satisfies (i) and (ii) of Lemma OA.10 for this multiplier. By (i), $T_W = M$ almost everywhere on $(\underline{\theta}, c)$, hence on $[\underline{\theta}, c]$ by continuity, so $W = F$ F-almost everywhere on $[\underline{\theta}, c]$, hence everywhere there, as $f > 0$ and W and F are continuous on $[\underline{\theta}, c]$. As $f > 0$ and $\kappa \in (c, \overline{\theta})$, $E'(s) = \int_c^s (t - \kappa) f(t)\, dt < 0$ on $(c, \kappa]$, so $E < 0$ there, and $E < 0$ on $(\kappa, \overline{\theta})$ by convexity; so by (ii), ν_W is carried by $(\underline{\theta}, c]$, W is affine on $[c, \overline{\theta}]$, and $W(c) = F(c)$ and

$W(\overline{\theta}) = 1$ (as $D > 0$) give $W = V$. So V is the unique solution, the same for every zero $c \in (\underline{\theta}, m)$ of ϕ, in particular for c^*.

□

Proposition OA.8. *Let F be single-peaked with mode m, $p > \frac{\theta+\overline{\theta}}{2}$ and F not convex. Let c^* be the smallest solution in $(\underline{\theta}, m]$ of $\mathbb{E}_F[\theta \mid \theta \geq c] = \frac{c+\overline{\theta}}{2}$, $q := F(c^*)$, and V^* the solution of (FRP).*

(i) Let $\kappa(\cdot \mid x)$, $x \in [c^*, \overline{\theta}]$, be a martingale kernel that spreads F_{c^*} into $U[c^*, \overline{\theta}]$: $\int S\, \kappa(dS \mid x) = x$ and $\int \kappa(\cdot \mid x)\, dF_{c^*}(x) = U[c^*, \overline{\theta}]$ (it exists by Strassen's theorem, as $F_{c^*} \leq_{cx} U[c^*, \overline{\theta}]$, and is supported in $[c^*, \overline{\theta}]$). The rule that selects the higher signal if $\min(x, y) < c^*$ and otherwise selects the agent with the higher independent score $S_x \sim \kappa(\cdot \mid x)$, $S_y \sim \kappa(\cdot \mid y)$ (coin at ties), has $V_F = V^*$, and full revelation is an equilibrium under it. A rule nondecreasing in the own signal with the same V_F exists: a lottery over coarse gradings whose cells are the singletons below c^* and a random partition of $[c^*, \overline{\theta}]$ into intervals.

(ii) No bar rule implements V^*. (For convex F with $p > \frac{\theta+\overline{\theta}}{2}$, $V^* = F$ is implemented by the noiseless rule, a degenerate pass mark.) (a) Lottery above a bar. *Under any rule that selects the higher signal unless both signals are at least B, and then tosses a coin, with B drawn independently and $\mathbb{P}(B < \overline{\theta}) > 0$, $V_F(x) \leq 1 - \frac{1}{2}\mathbb{P}(\theta \geq B,\, x \geq B)$, so $V_F(\overline{\theta}-) < 1 = V^*(\overline{\theta})$; this holds for every prior. (b)* Pass mark above c^*. *Under any rule that selects the higher signal unless both signals lie in $[c^*, B)$, and then tosses a coin, with B independent and $\mathbb{P}(B > c^*) > 0$, $\liminf_{x \downarrow c^*} V_F(x) - \lim_{x \uparrow c^*} V_F(x) \geq \frac{1}{2}\mathbb{P}\big(\theta \in [c^*, B)\big) > 0$, so V_F is not convex and full revelation is not an equilibrium. (c) Under a random pass mark (the higher signal is selected if it is at least an independent bar B, and a coin is tossed otherwise), $V_F = F$ on $(\underline{\theta}, c^*)$ only if $B \leq \underline{\theta}$ a.s.; then the rule is noiseless and $V_F = F \neq V^*$.*

Proof. (i) Against a fully revealing rival, a signal $x < c^*$ wins if and only if $y < x$, so $V_F(x) = F(x)$; a signal $x \geq c^*$ wins against every $y < c^*$ and, conditionally on $y \geq c^*$, with probability $\mathbb{P}(S_y < S_x) = \mathbb{E}[U_{c^*}(S_x)] = U_{c^*}(x)$, as $S_y \sim U[c^*, \overline{\theta}]$ unconditionally and U_{c^*} is affine on $[c^*, \overline{\theta}]$, which contains the support of S_x. So $V_F(x) = q + (1-q)\frac{x-c^*}{\overline{\theta}-c^*} = V^*(x)$. As V^* is convex on $[\underline{\theta}, \overline{\theta}]$, full revelation is a best reply to itself by Jensen's inequality. For the monotone implementation, apply Lemma 17(i) to the prior F_{c^*} on $[c^*, \overline{\theta}]$ and the nondecreasing feasible value function U_{c^*}: it gives a rule $\tilde{H}$ on $[c^*, \overline{\theta}]^2$, nondecreasing in the own signal, with $\int \tilde{H}(x, y)\, dF_{c^*}(y) = U_{c^*}(x)$. Let $H(x, y) := \mathbb{1}\{x > y\} + \frac{1}{2}\mathbb{1}\{x = y\}$ if $\min(x, y) < c^*$ and $H := \tilde{H}$ if both signals are

at least c^*. For fixed $y < c^*$, $x \mapsto H(x, y)$ is nondecreasing; for fixed $y \geq c^*$ it is 0 below c^* and nondecreasing and nonnegative above, so H is nondecreasing in the own signal, and against a fully revealing rival $V_F(x) = F(x)$ for $x < c^*$ and $V_F(x) = q + (1-q)U_{c^*}(x)$ for $x \geq c^*$, i.e. $V_F = V^*$. (ii)(a) A signal x loses the coin with probability $\frac{1}{2}$ whenever $y \geq B$ and $x \geq B$, so $V_F(x) \leq 1 - \frac{1}{2}\mathbb{P}(\theta \geq B, x \geq B) \to 1 - \frac{1}{2}\mathbb{P}(\theta \geq B) < 1$ as $x \uparrow \overline{\theta}$, since $\mathbb{P}(\theta \geq B) = \mathbb{E}[1 - F(B^-)] > 0$ when $\mathbb{P}(B < \overline{\theta}) > 0$. (b) For $x < c^*$, $V_F(x) = F(x) \to F(c^*)$; for $x > c^*$, x beats every $y < c^*$ and, when $B > x$, gets the coin against every $y \in [c^*, B)$, so $V_F(x) \geq F(c^*) + \frac{1}{2}\mathbb{P}(\theta \in [c^*, B),\ B > x)$, which tends to $F(c^*) + \frac{1}{2}\mathbb{P}(\theta \in [c^*, B))$ as $x \downarrow c^*$ by monotone convergence; this exceeds $F(c^*)$ as $f > 0$ and $\mathbb{P}(B > c^*) > 0$. A convex function is continuous on $(\underline{\theta}, \overline{\theta})$, so V_F is not convex; it is upper semicontinuous and $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$, so by Lemma 13 full revelation is not an equilibrium. (c) Let K be the CDF of B. As in the proof of Lemma OA.6, $V_F(x) = \frac{1}{2} + \frac{1}{2}K(x)F(x) - \frac{1}{2}\int_x^{\overline{\theta}} K\, dF$, so $d(F^2K) = 2F\, dV_F$; if $V_F = F$ on $(\underline{\theta}, c^*)$, then $d(F^2K) = d(F^2)$ there, so $F^2K = F^2 + C$ on $(\underline{\theta}, c^*)$ for a constant C. As $x \downarrow \underline{\theta}$, $F(x) \to 0$ and $K \in [0, 1]$, so $C = 0$; hence $K \equiv 1$ on $(\underline{\theta}, c^*)$ and $B \leq \underline{\theta}$ a.s. Then the higher signal is never below B (up to a null event), the rule is noiseless, and $V_F = F$ on $[\underline{\theta}, \overline{\theta}]$, which differs from V^* because V^* is convex and F is not. □

OA.3.6 The random pass mark

A randomly drawn pass mark implements the uniformizing rule of a given width if and only if $\sqrt{I_F}$ is concave on the revealed bottom $[\underline{\theta}, t_1]$; for concave priors, it implements the whole family, and its equilibrium is unique.

Lemma OA.11. *Let F satisfy (i)–(iii) of Theorem 3 and have no atom in $(\underline{\theta}, \overline{\theta}]$, let $\bar{s} \in [2p - \underline{\theta}, \overline{\theta}]$, $G^* := G^*_{\bar{s}}$, and let $t_1 \leq t_2$ be as in Proposition 12, with $t_1 = t_2 := \underline{\theta}$ if $\bar{s} = 2p - \underline{\theta}$ and $t_1 = t_2 := \overline{\theta}$ if $\bar{s} = \overline{\theta}$. For $z > \underline{\theta}$ let*

$$R_{\bar{s}}(z) := \frac{I_{G^*}(z)}{(\bar{s} - p)\, G^*(z)^2}, \qquad K_{\bar{s}}(z) := \min\{1, R_{\bar{s}}(z)\},$$

$K_{\bar{s}}(\underline{\theta}) := K_{\bar{s}}(\underline{\theta}^+)$ and $K_{\bar{s}} := 0$ below $\underline{\theta}$. Draw a pass mark C independently of the signals with $\mathbb{P}(C \leq z) = K_{\bar{s}}(z)$, and select the higher signal if it is at least C, tossing a coin otherwise.

(i) $R_{\bar{s}}$ is nondecreasing on $(\underline{\theta}, \infty)$ if and only if $\sqrt{I_{G^*}}$ is concave on $[\underline{\theta}, \infty)$, if and only if $\sqrt{I_F}$ is concave on $[\underline{\theta}, t_1]$. For $\bar{s} < \overline{\theta}$ this is also equivalent to $K_{\bar{s}}$ being nondecreasing. When it holds, $K_{\bar{s}} = R_{\bar{s}} \leq 1$ on $(\underline{\theta}, \bar{s}]$ and $K_{\bar{s}} = 1$ on $[t_2, \infty)$, so $K_{\bar{s}}$ is a CDF.

(ii) If the condition in (i) holds, then (G^*, G^*) is an equilibrium in which the agent's value function is $L_{\bar{s}}$ on $[\underline{\theta}, \bar{s}]$ and 1 above $\bar{s}$, so the rule attains $u_F(\bar{s})$.

(iii) Conversely, if some random pass mark, with any CDF $\tilde{K}$, gives a value function against G^* equal to $L_{\bar{s}}$ G^*-almost everywhere, then the condition in (i) holds and $\tilde{K} = K_{\bar{s}}$ G^*-almost everywhere; indeed $V_{G^*} - L_{\bar{s}} = \frac{1}{2}G^*(\tilde{K} - R_{\bar{s}})$ on $(\underline{\theta}, \bar{s}]$.

(iv) If the condition in (i) holds, $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, and F is strictly increasing on $[\underline{\theta}, t_1]$, then (G^*, G^*) is the unique equilibrium. For non-uniform concave F, Corollary 3 gives uniqueness at $\bar{s} = 2p - \underline{\theta}$, where the rule is noiseless.

So the uniformizing rule of width $\bar{s}$ is implemented by a random pass mark if and only if $\sqrt{I_F}$ is concave on $[\underline{\theta}, t_1]$, equivalently I_F/F^2 is nondecreasing on $(\underline{\theta}, t_1]$, equivalently, if F has a density f there, $F^2 \geq 2fI_F$ almost everywhere on $(\underline{\theta}, t_1]$. This holds for every F that is concave on $[\underline{\theta}, t_1]$, but fails if F has a density that is strictly increasing at $\underline{\theta}$.

Proof. Throughout, $G := G^*$ unless a rival G is named. Under a pass mark with CDF K, $H(x, y) = \frac{1}{2} + K(\max(x, y))\big(\mathbb{1}\{x > y\} - \frac{1}{2}\big)$, which is nondecreasing in x, and a tie contributes $\frac{1}{2}$, so

$$V_G(x) = \tfrac{1}{2} + \tfrac{1}{2}K(x)\,G(x^-) - \tfrac{1}{2}\int_{(x,\infty)} K\,dG. \tag{OA.6}$$

Preliminaries. (P0) For $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, $t_1 < t_2$ and $I_{G^*} < I_F$ on $(t_1, \overline{\theta})$. With $D := I_F - I_{\bar{U}_{\bar{s}}}$ as in the proof of Proposition 12, $D' = F - L_{\bar{s}}$ on $[\underline{\theta}, \bar{s}]$ is positive exactly on an interval (a, b) by (iii), and $D' \leq 0$ elsewhere. As $D < 0$ just left of its crossing x^* and $D > 0$ just right of it, $x^* \in (a, b)$: if $x^* \leq a$, D would be nonincreasing on $(-\infty, x^*]$, so $D \geq D(x^*) = 0$ there; if $x^* \geq b$, D would be nonincreasing on $[x^*, \overline{\theta}]$, so $D \leq 0$ just right of x^*. Hence the slope of I_F at x^* exceeds that of $I_{\bar{U}_{\bar{s}}}$, $\min(I_F, I_{\bar{U}_{\bar{s}}})$ has a concave kink at x^*, and its convex minorant lies strictly below it near x^*, so $t_1 < x^* < t_2$, and $I_{G^*} < \min(I_F, I_{\bar{U}_{\bar{s}}}) \leq I_F$ on (t_1, t_2). On $(x^*, \overline{\theta})$, $D > 0$: D increases on (x^*, b) and is nonincreasing on $(b, \overline{\theta})$ with $D(\overline{\theta}) = 0$, and $D = 0$ on some $[c, \overline{\theta})$ with $c < \overline{\theta}$ would force $F = 1$ on $[\max(c, \bar{s}), \overline{\theta})$, contradicting $\overline{\theta} = \max \operatorname{supp}(F)$. So $I_{G^*} = I_{\bar{U}_{\bar{s}}} < I_F$ on $[t_2, \overline{\theta})$, which with the above gives $I_{G^*} < I_F$ on $(t_1, \overline{\theta})$.

(P1) $I_{G^*} > 0$ on $(\underline{\theta}, \infty)$, so $G^* > 0$ there. If $t_1 > \underline{\theta}$, $G^* = F$ on $[\underline{\theta}, t_1)$ and $\underline{\theta} = \min \operatorname{supp}(F)$; if $t_1 = \underline{\theta}$ for an interior width, the bridge has slope $m_R := G^*(t_1) = \bar{U}_{\bar{s}}(t_2) > 0$ and touches I_F at $\underline{\theta}$, which forces an atom of F at $\underline{\theta}$, as otherwise $I_F(x) = o(x - \underline{\theta})$ lies below the bridge just right of $\underline{\theta}$; and G^* keeps mass m_R there.

(P2) For a continuous convex I with right derivative G and $I > 0$ on an open interval J, $\sqrt{I}$ is concave on J if and only if its right derivative $G/(2\sqrt{I})$ is nonincreasing on J, if and only if I/G^2 is nondecreasing on J. If G has a density g on J, $G/\sqrt{I}$ is locally absolutely continuous with derivative $(gI - \frac{1}{2}G^2)/I^{3/2}$ almost everywhere, so the condition is $G^2 \geq 2gI$ almost everywhere on J.

(P3) Let $\bar{s}$ be interior and $d := \frac{1}{2(\bar{s}-p)}$. On $[t_1, t_2)$, $G^* = m_R$ and $I_{G^*}(z) = I_F(t_1) + m_R(z - t_1)$, so $R_{\bar{s}}$ is strictly increasing there; on $[t_2, \bar{s}]$, $I_{G^*} = \frac{d}{2}(z - 2p + \bar{s})^2$ and $G^* = d(z - 2p + \bar{s})$, so $R_{\bar{s}} = 1$; above $\bar{s}$, $R_{\bar{s}} = \frac{z-p}{\bar{s}-p} > 1$. As I_{G^*} is continuous and $G^*(t_2^-) = m_R$, $R_{\bar{s}}(t_2^-) = 1$, so $R_{\bar{s}} < 1$ on $[t_1, t_2)$, where it is strictly increasing.

(i) By (P1) and (P2) with $J = (\underline{\theta}, \infty)$, $R_{\bar{s}}$ is nondecreasing if and only if $\sqrt{I_{G^*}}$ is concave on $(\underline{\theta}, \infty)$, hence on $[\underline{\theta}, \infty)$ by continuity. The right derivative $G^*/(2\sqrt{I_{G^*}})$ is nonincreasing on each of $[t_1, t_2)$, $[t_2, \bar{s})$, where it is constant, and $[\bar{s}, \infty)$, and it is continuous at t_1, t_2 and $\bar{s}$, as F has no atom at t_1; so it is nonincreasing on $(\underline{\theta}, \infty)$ if and only if it is nonincreasing on $(\underline{\theta}, t_1)$, i.e., $\sqrt{I_F}$ is concave on $[\underline{\theta}, t_1]$, as $I_{G^*} = I_F$ there. At the ends the same argument applies with $[\underline{\theta}, t_1] = [\underline{\theta}, \underline{\theta}]$, where $G^* = \bar{U}_{2p-\underline{\theta}}$ and $R_{\bar{s}} \equiv 1$ on $(\underline{\theta}, \bar{s}]$, and with $[\underline{\theta}, t_1] = [\underline{\theta}, \overline{\theta}]$, where $G^* = F$. If $R_{\bar{s}}$ is nondecreasing, then $R_{\bar{s}} \leq R_{\bar{s}}(t_2) = 1$ on $(\underline{\theta}, t_2]$, so $K_{\bar{s}} = R_{\bar{s}}$ there and $K_{\bar{s}} = 1$ above; it is continuous on $(\underline{\theta}, \infty)$, as I_{G^*} and $G^* > 0$ are, so it is a CDF. Conversely, for $\bar{s} < \overline{\theta}$, if $K_{\bar{s}}$ is nondecreasing, then $K_{\bar{s}}(z) \leq K_{\bar{s}}(t_1) = R_{\bar{s}}(t_1) < 1$ for $z \leq t_1$ by (P3), so $K_{\bar{s}} = R_{\bar{s}}$ on $(\underline{\theta}, t_2)$, and $R_{\bar{s}}$ is nondecreasing on $(\underline{\theta}, t_2]$, hence on $(\underline{\theta}, \infty)$ by (P3).

(ii) Let the condition hold, and let $K := K_{\bar{s}}$. Then G is continuous on $(\underline{\theta}, \infty)$ and $KG^2 = 2d\, I_G$ on $(\underline{\theta}, \bar{s}]$. As measures on $(\underline{\theta}, \bar{s}]$, since G^2 is continuous and K is of bounded variation, $d(KG^2) = G^2 dK + 2KG\, dG$, while $d(2dI_G) = 2d\, G\, dz$; dividing by $G > 0$, $G\, dK + 2K\, dG = 2d\, dz$. By (OA.6), $dV_G = \frac{1}{2}(K\, dG + G\, dK) + \frac{1}{2}K\, dG = d\, dz$. So V_G is affine with slope d on $(\underline{\theta}, \bar{s}]$, and $V_G(\bar{s}) = \frac{1}{2} + \frac{1}{2}K(\bar{s}) = 1 = L_{\bar{s}}(\bar{s})$, as G puts no mass above $\bar{s}$; hence $V_G = L_{\bar{s}}$ on $(\underline{\theta}, \bar{s}]$. At $\underline{\theta}$: if G has no atom there, $V_G(\underline{\theta}^+) - V_G(\underline{\theta}) = \lim_{x \downarrow \underline{\theta}} \frac{1}{2}K(x)G(x) = 0$; if G has an atom a at $\underline{\theta}$, the gap is $\frac{1}{2}K(\underline{\theta}^+)a$, and $K(\underline{\theta}^+) \leq \lim_{x \downarrow \underline{\theta}}(x - \underline{\theta})/((\bar{s} - p)a^2) = 0$, as $I_G(x) \leq x - \underline{\theta}$. So $V_G = L_{\bar{s}}$ on $[\underline{\theta}, \bar{s}]$, while $V_G = \frac{1}{2} + \frac{1}{2}K = 1 < L_{\bar{s}}$ on $(\bar{s}, \overline{\theta}]$. Hence every feasible G' gives $\int V_G\, dG' \leq \int L_{\bar{s}}\, dG' = \frac{1}{2} = \int V_G\, dG$, so (G^*, G^*) is an equilibrium, and by Lemma 14, $u_F = p + 2\,\mathrm{Cov}_{G^*}(s, L_{\bar{s}}(s)) = p + \mathrm{Var}(G^*)/(\bar{s} - p) = u_F(\bar{s})$.

(iii) G is continuous on $(\underline{\theta}, \bar{s}]$. Let $\tilde{K}$ be any CDF, $V := V_G$ under it, $W := V - L_{\bar{s}}$ and $\Psi := \tilde{K}G^2 - 2d\, I_G$ on $(\underline{\theta}, \bar{s}]$. As in (ii), $d\Psi = 2G\, dV - 2d\, G\, dz = 2G\, dW$, and, integrating by parts with G continuous, $d(\Psi - 2GW) = -2W\, dG$. As $W = 0$ G-almost everywhere, $\Psi - 2GW$ is constant on $(\underline{\theta}, \bar{s}]$. Its limit at $\underline{\theta}^+$ is 0: without

an atom at $\underline{\theta}$, $G(\underline{\theta}^+) = I_G(\underline{\theta}^+) = 0$ and W is bounded; with an atom a, by (OA.6), $V(\underline{\theta}^+) - V(\underline{\theta}) = \frac{1}{2}\tilde{K}(\underline{\theta})a$ and $V(\underline{\theta}) = L_{\bar{s}}(\underline{\theta})$, so $\Psi(\underline{\theta}^+) - 2aW(\underline{\theta}^+) = \tilde{K}(\underline{\theta})a^2 - \tilde{K}(\underline{\theta})a^2 = 0$. Hence $\Psi = 2GW$, i.e., $\tilde{K} - R_{\bar{s}} = 2W/G$ on $(\underline{\theta}, \bar{s}]$, which is the stated identity, and $\tilde{K} = R_{\bar{s}}$ G-almost everywhere on $[\underline{\theta}, \bar{s}]$. Let $S := \mathrm{supp}(G) \cap (\underline{\theta}, t_1)$ and $N \subseteq S$ the G-null set where $\tilde{K} \neq R_{\bar{s}}$. $S \setminus N$ is dense in S, $R_{\bar{s}}$ is continuous on $(\underline{\theta}, t_1]$ and $\tilde{K}$ is nondecreasing, so $R_{\bar{s}}$ is nondecreasing on S; on each component of $(\underline{\theta}, t_1) \setminus S$, G is constant and I_G affine and increasing, so $R_{\bar{s}}$ is increasing there, and the components' endpoints lie in $S \cup \{\underline{\theta}, t_1\}$. Hence $R_{\bar{s}}$ is nondecreasing on $(\underline{\theta}, t_1]$, so $\sqrt{I_F}$ is concave on $[\underline{\theta}, t_1]$ by (P2), and (i) gives the condition. Then $\tilde{K} = R_{\bar{s}} = K_{\bar{s}}$ G-almost everywhere.

(iv) Let $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, so that F is not uniform, and write $K := K_{\bar{s}}$, which by (i) and (P1) is continuous, nondecreasing and positive on $(\underline{\theta}, \infty)$. If (G, G') is an equilibrium, then by interchangeability so is (G, G^*), so G is a best reply to G^* and G^* is a best reply to G. The first gives $\mathrm{supp}(G) \subseteq [\underline{\theta}, \bar{s}]$. The second is used through deviations from G^* that remain feasible: any mean-preserving contraction of G^*, and, as $I_{G^*} < I_F$ on $(t_1, \overline{\theta})$ by (P0), any small mean-preserving perturbation of G^* compactly supported there.

Step 1. Two-sided perturbations inside $(t_2, \bar{s})$ make V_G affine there. As $K = 1$ there, V_G is the mid-CDF of G, so G has a constant density d' on $(t_2, \bar{s})$ and no atoms there. Spreading mass from near $\bar{s}$ to just above it, where $V_G = 1$, is unprofitable only if G has no atom at $\bar{s}$. So $V_G = \ell$ on $(t_2, \bar{s})$ with $\ell(x) := 1 - d'(\bar{s} - x)$.

Step 2. G has no atom in $(\underline{\theta}, t_1)$. At an atom a, $V_G(a^+) - V_G(a^-) = K(a)G(\{a\}) > 0$ by (OA.6) and continuity of K, and V_G is nondecreasing. As F is strictly increasing on $[\underline{\theta}, t_1]$, $G^* = F$ puts mass on $[a - \eta, a)$ and on $[a + \eta'/2, a + \eta']$ for all small $\eta, \eta' > 0$. Fix η' and pool all of G^* on $[a - \eta, a)$ with a fraction of G^* on $[a + \eta'/2, a + \eta']$ into their mean $m \in (a, a + \eta)$, chosen off the atoms of G; this is a contraction of G^*, and since the left part has first moment about m at most $2\eta\, G^*([a - \eta, a))$ while each right type lies at distance at least $\eta'/4$ from m, the fraction used carries mass at most $\frac{8\eta}{\eta'} G^*([a - \eta, a))$. The pooled types below a gain at least $V_G(a^+) - V_G(a - \eta) \to K(a)G(\{a\})$ each as $\eta \to 0$, while the others lose at most 1 each, so the deviation pays for small η. Hence G and V_G are continuous on $(\underline{\theta}, t_1)$. Merging a small mass of G^* at a signal $u < t_2$ with pooled mass at $x \in (t_2, \bar{s})$ into a point $y \in (t_2, x)$ is a contraction of G^* and changes the payoff by a positive multiple of $\ell(u) - V_G(u)$. Hence $V_G \geq \ell$ G^*-almost everywhere below t_2.

Step 3. If $d' > d$, then $G < G^*$ on $(t_2, \bar{s})$, and $G \leq 1 - d'(\bar{s} - t_2) < m_R = G^*$ on (t_1, t_2), while $\int_{\underline{\theta}}^{t_1} (G - G^*)\, dx = I_G(t_1) - I_F(t_1) \leq 0$. So $\int (G - G^*)\, dx < 0$, contradicting equal means. Hence $d' \leq d$.

Step 4. The equilibrium value gives $\frac{1}{2} = \int V_G\, dG^* \geq \int \ell\, dG^* = \ell(p) = 1 - \frac{d'}{2d}$, by

Steps 1 and 2. Hence $d' = d$, $\ell = L_{\bar{s}}$, $G = G^*$ on $(t_2, \bar{s}]$, and, as the inequality is now an equality, $V_G = L_{\bar{s}}$ G^*-almost everywhere below t_2; as V_G and $L_{\bar{s}}$ are continuous on $(\underline{\theta}, t_1)$ and $\operatorname{supp}(G^*) \supseteq [\underline{\theta}, t_1]$, $V_G = L_{\bar{s}}$ on $(\underline{\theta}, t_1)$, and also at $\underline{\theta}$ if $\underline{\theta}$ is an atom of G^*.

Step 5. If $t_1 = \underline{\theta}$, then G^* has an atom at $\underline{\theta}$, and $V_G(\underline{\theta}) = L_{\bar{s}}(\underline{\theta}) = V_{G^*}(\underline{\theta})$ with $G = G^*$ above t_2 gives $\int_{(\underline{\theta}, t_2]} K\, dG = 0$, so G has no mass on $(\underline{\theta}, t_2]$ and $G = G^*$. Otherwise, letting $x \uparrow t_1$ in $V_G(x) = L_{\bar{s}}(x) = V_{G^*}(x)$, with $G = G^*$ above t_2 and G^* putting no mass on $[t_1, t_2]$, gives $K(t_1)\big(G(t_1^-) - m_R\big) = \int_{[t_1, t_2]} K\, dG \geq K(t_1)\, G([t_1, t_2]) = K(t_1)\big(m_R - G(t_1^-)\big)$, so $G(t_1^-) \geq m_R$; as $G(t_1^-) \leq G(t_2) = m_R$, G has no mass on $[t_1, t_2]$. On $[\underline{\theta}, t_1)$, with $E := G - F$, the same comparison and an integration by parts give $2K(x)E(x) = -\int_x^{t_1} E\, dK =: -\Phi(x)$. For every $x_0 \in (\underline{\theta}, t_1)$, $K \geq K(x_0) > 0$ on $[x_0, t_1]$, so $|\Phi(x)| \leq \int_x^{t_1} \frac{|\Phi|}{2K(x_0)}\, dK$ there, and $\Phi \equiv 0$ on $[x_0, t_1]$ by Grönwall's inequality. So $E \equiv 0$ on $(\underline{\theta}, t_1)$, and also at $\underline{\theta}$ by right-continuity. Hence $G = G^*$, and every equilibrium is (G^*, G^*). □

OA.3.7 Continuous rules

Lemma OA.12. *Let F be concave and not uniform, let $\bar{s}^*$ be the best width, and let K be the pass-mark CDF of Lemma OA.11 at $\bar{s}^*$, restricted to signals in $[\underline{\theta}, \overline{\theta}]$. For $\varepsilon > 0$, let*

$$H^\varepsilon(x, y) := \tfrac{1}{2} + K(\max(x, y))\, \rho\Big(\frac{x - y}{\varepsilon}\Big), \qquad \rho(t) := \max\{-\tfrac{1}{2}, \min\{\tfrac{t}{2}, \tfrac{1}{2}\}\}.$$

H^ε is continuous and equals the pass-mark rule when $|x - y| \geq \varepsilon$. It has an equilibrium, and as $\varepsilon \to 0$, every equilibrium strategy of H^ε converges weakly to $G^*_{\bar{s}^*}$, and the firm's payoff in every equilibrium converges to $u_F(\bar{s}^*)$. Likewise, if F is convex and atomless with $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$ *and $K \equiv 1$, every equilibrium strategy converges weakly to F, and the firm's payoff converges to the first best.*

Proof. Write H for the pass-mark rule and $G^* := G^*_{\bar{s}^*}$. K, with $K(\underline{\theta}) := K(\underline{\theta}^+)$, is continuous on $[\underline{\theta}, \overline{\theta}]$, so H^ε is continuous, and an equilibrium exists by Sion's theorem; each equilibrium strategy guarantees $\frac{1}{2}$. H is continuous off the diagonal, $H^\varepsilon \to H$ uniformly off every neighborhood of the diagonal, and at a diagonal point (y, y) all limits of H^ε lie between $\frac{1}{2} - \frac{1}{2}K(y)$ and $\frac{1}{2} + \frac{1}{2}K(y)$. Hence, if $\mu_\varepsilon \to \mu$ weakly and μ puts no mass on the diagonal, $\int H^\varepsilon\, d\mu_\varepsilon \to \int H\, d\mu$; the same holds if μ puts mass on the diagonal only at $(\underline{\theta}, \underline{\theta})$ and $K(\underline{\theta}^+) = 0$.

Let G_ε be equilibrium strategies; the feasible set is weakly compact, so take a convergent subsequence, $G_\varepsilon \to G$. (i) G is a best reply to G^* under H: $\iint H^\varepsilon\, dG_\varepsilon\, dG^* \geq \frac{1}{2}$, and $G \otimes G^*$ puts mass on the diagonal only where G^* has an atom, which is at

most at $\underline{\theta}$, where $K(\underline{\theta}^+) = 0$; so $\int V_{G^*}\, dG \geq \frac{1}{2}$. (ii) G^* is a best reply to G under H against every deviation G' that has no atom in common with G except possibly at $\underline{\theta}$: $\iint H^\varepsilon\, dG'\, dG_\varepsilon \leq \frac{1}{2}$, and $G' \otimes G$ puts no mass on the diagonal, so $\int V_G\, dG' \leq \frac{1}{2}$, while $\int V_G\, dG^* = \frac{1}{2}$. The uniqueness part of the proof of Lemma OA.11 uses only (i) and (ii): every deviation in its Steps 1 and 2 can avoid the countably many atoms of G other than $\underline{\theta}$. Hence $G = G^*$, and the whole family converges to G^*. The firm's payoff is $\iint c_\varepsilon\, dG_\varepsilon\, dG'_\varepsilon$ with $c_\varepsilon(x, y) := y + (x - y)H^\varepsilon(x, y)$, which converges uniformly, as $|c_\varepsilon - c| \leq \varepsilon$, to $c(x, y) := y + (x - y)H(x, y)$, a continuous function; so the payoff converges to $u_F(\bar{s}^*)$.

For convex F, (i) and (ii) hold with F in place of G^*, as full revelation is an equilibrium of the noiseless rule and F is atomless; $V_G(x) = G(x^-) + \frac{1}{2}G(\{x\})$ is the mid-CDF. G has no atom: it has none at $\underline{\theta}$ or $\overline{\theta}$, as $G \leq_{cx} F$ and F is atomless, and at an interior atom x_0, V_G jumps up, and pooling the types near x_0 into a point just above x_0 that is not an atom of G would pay. So $V_G = G$ is continuous, and every $G' \leq_{cx} F$ is admissible in (ii). Pooling the types in an interval at their mean creates a single atom, which G does not have, and F is continuous and strictly increasing on $[\underline{\theta}, \overline{\theta}] = \operatorname{supp}(F)$. So G is convex on $[\underline{\theta}, \overline{\theta}]$ by Lemma 12 with $K = F$ and $h = G$. By (i), (ii) and constant-sum, $\int G\, dF = \int F\, dG = \frac{1}{2} = \int F\, dF = \int G\, dG$, so $\int F\, d(F - G) = \int G\, d(F - G) = 0$, and the last paragraph of Appendix C.8.3, with $m = 0$ and $\hat{G} = G$, gives $G = F$, and the payoff converges to $\mathbb{E}\max(\theta_1, \theta_2)$. □

OA.3.8 The noiseless equilibrium

Lemma OA.13. *Let (G, G) be a symmetric equilibrium of the noiseless rule, with F, c and E as in Lemma 20. If (a, b) is a component of $(\underline{\theta}, c) \setminus E$ and a is a limit of points of E from the left, the slope of G on (a, b) is $f(a)$.*

Proof. Let γ be the slope of G on (a, b). As $I_G \leq I_F$ with equality at a and $G(a) = F(a)$, $\gamma\frac{(x-a)^2}{2} \leq \int_a^x (F - F(a))\, dt$ for $x \in (a, b)$, so $\gamma \leq f(a)$. If $x_n \uparrow a$ with $x_n \in E$, the left derivative of G at a is $\lim \frac{F(a)-F(x_n)}{a-x_n} = f(a)$, and convexity gives $\gamma \geq f(a)$. □

Corollary OA.2. *The noiseless equilibrium of every F with a non-constant density that is real-analytic on an open interval containing $[\underline{\theta}, \overline{\theta}]$ is well behaved.*

Proof. If f is real-analytic on an open interval containing $[\underline{\theta}, \overline{\theta}]$ and not constant, f' has finitely many zeros in $[\underline{\theta}, \overline{\theta}]$, as otherwise they would accumulate and $f' \equiv 0$ by the identity theorem, and f is strictly monotone between them. So Proposition 14 applies. □

Proposition OA.9. *Every Beta(a, b) prior other than the uniform has a well-behaved noiseless equilibrium.*

Proof. If $a, b \geq 1$, this follows from Proposition 14 with $m \leq 2$, as f is continuous on $[0,1]$ and strictly monotone on each side of its mode. If $a \geq 1 > b$, f is strictly increasing, so $G_0 = F$ by Lemma 20, and $[0,1]$ is a revealed stretch, as F is strictly convex and twice continuously differentiable on $(0,1)$. Otherwise $a < 1$, so $f(0^+) = \infty$, and, for some $z \in (0,1]$, f is strictly decreasing on $(0,z]$ and strictly increasing on $[z,1)$; so, for every line L through the origin and every $t_1 \in (0,1]$ with $F(t_1) \leq L(t_1)$, $F - L$ is positive and then nonpositive on $(0,t_1]$, being positive near 0, concave on $[0,z]$ and convex on $[z,1]$. If $p \leq \frac{1}{2}$, let $G_0 := U[0,2p]$ and $L(t) := \frac{t}{2p}$. Then $I_F - I_{G_0}$ vanishes at 0, rises and then falls on $[0,2p]$, and equals $\int_t^1 (1-F)\,dx \geq 0$ at every $t \geq 2p$; so $G_0 \leq_{cx} F$, G_0 satisfies (a)–(d) of Lemma 20, $[0,2p]$ is a single block, and (H3) holds as $f \to \infty$ at 0, and at 1 if $p = \frac{1}{2}$, as then $a = b$. If $p > \frac{1}{2}$, then $b < a < 1$. Let $\varphi(t) := I_F(t) - \frac{t}{2}F(t)$. As $(F - tf)' = -tf'$ and $tf(t) \to 0$ as $t \to 0$, $\varphi' = \frac{1}{2}(F - tf)$ is positive on $(0,z]$, strictly decreasing on $(z,1)$, and tends to $-\infty$ at 1; as $\varphi(0) = 0 > \frac{1}{2} - p = \varphi(1)$, φ has a unique zero $t^* \in (0,1)$, and $t^* > z$ and $\varphi'(t^*) < 0$, i.e., $\gamma := F(t^*)/t^* < f(t^*)$. Let $G_0(t) := \gamma t$ on $[0,t^*]$ and $G_0 := F$ on $[t^*,1]$. It is convex, as F is convex on $[z,1]$ and $\gamma < f(t^*)$; with $L(t) := \gamma t$ and $t_1 := t^*$, $I_F - I_{G_0}$ vanishes at 0, rises and then falls to $\varphi(t^*) = 0$ on $[0,t^*]$, and vanishes on $[t^*,1]$. So G_0 satisfies (a)–(d) of Lemma 20 with $c = 1$ and $E_0 = \{0\} \cup [t^*,1]$; $[0,t^*]$ is a block, $[t^*,1]$ is a revealed stretch, and (H3) holds as $f \to \infty$ at 0 and $f(t^*) > \gamma$. □

The conclusion of Theorem 4 holds whenever (H1) holds and (H2) and (H3) hold on one chain and the revealed stretches next to it, as the proof uses them only there.

OA.3.9 Weakening condition (ii) of Theorem 3

Condition (ii) enters the proof of Theorem 3 only at widths $\bar{s} \geq \overline{\theta}$, through the identity $\mathrm{Var}(G^*_{\bar{s}}) = \mathrm{Var}(F)$ in Step 5: Steps 1–4 and 6 use none of (i)–(iii), and Step 5 uses (i) only for $\bar{s} \leq 2p - \underline{\theta}$ and (iii) only for $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$. This subsection replaces (ii) by a condition on finitely many numbers, using that Step 5's functions, built from any pair $t_1 < t_2$, stay admissible at every width.

Lemma OA.14. *Let $t_1 < t_2$ be any reals, and let C, w_1, w_2 and a be as in Step 5 of the proof of Theorem 3, with these t_1, t_2. Then w_1, w_2 are convex, a is concave, and $R(s,y) \geq 0$ for all real*

s, y. Let $K := \int w_1\, dF$ and $q(h) := \frac{h^3}{6} + \frac{(t_2-t_1)h^2}{4}$. If (G, G), s, Y and $\bar{s}$ are as in Step 4, with $\bar{s} \geq t_2$, then

$$\begin{aligned} \mathbb{E}[(Y-p)(s-p)] \leq K + \int w_2\, d\bar{U}_{\bar{s}} \leq K + \frac{q(\bar{s}-t_2)}{\bar{s}-p}, \quad \text{so} \\ u_F \leq B(\bar{s}) := p + \frac{K}{\bar{s}-p} + \frac{q(\bar{s}-t_2)}{(\bar{s}-p)^2}, \end{aligned} \tag{OA.7}$$

where the second inequality is an equality if $t_2 \geq 2p - \bar{s}$.

Proof. Here $(x-p)^2 - C(x) = (x-t_1)(x-t_2)$. The slope of w_1 jumps at t_1 from $2(t_1-p)$ to t_1+t_2-2p, i.e., up by $t_2-t_1>0$; w_2 is 0 and then a convex quadratic with slope $t_2-t_1>0$ at t_2; and a is a minimum of affine functions. As $y-p \leq t_1-p$ if and only if $y \leq t_1$, and $t_1+t_2-p-y \leq t_1-p$ if and only if $y \geq t_2$, $a(y)$ equals $y-p$ for $y \leq t_1$, t_1-p for $t_1 \leq y \leq t_2$, and t_1+t_2-p-y for $y \geq t_2$. Expanding, $R(s,y)$ equals

	$s \leq t_1$	$s \geq t_1$
$y \leq t_1$:	$(s-y)^2$	$(t_1-y)^2 + (s-t_1)(t_1+t_2-2y)$
$t_1 < y < t_2$:	$(y-s)(t_1-s)$	$(s-t_1)(t_2-y)$
$y \geq t_2$:	$(t_1-s)(t_2-s)$	0

and each expression is nonnegative on its region; none involves p, $\bar{s}$ or the position of t_1, t_2 relative to the prior. As in Step 5, $G \leq_{cx} F$ and $Y \leq_{cx} \bar{U}_{\bar{s}}$ (Step 4) give $\mathbb{E}w_1(s) \leq K$ and $\mathbb{E}w_2(Y) \leq \int w_2\, d\bar{U}_{\bar{s}}$, and Step 6, which holds for every concave a, gives $\mathbb{E}[a(Y)(s-Y)] \geq 0$; this is the first inequality in (OA.7). Next, $\int w_2\, d\bar{U}_{\bar{s}}$ integrates $(x-t_1)(x-t_2)$ against the density $1/(2(\bar{s}-p))$ over $[\max(t_2, 2p-\bar{s}), \bar{s}]$, the integrand is nonnegative on $[t_2, \infty)$, and $\int_{t_2}^{\bar{s}} (x-t_1)(x-t_2)\, dx = 2q(\bar{s}-t_2)$. Finally, $u_F = p + \mathbb{E}[(Y-p)(s-p)]/(\bar{s}-p)$ by Step 4. □

At the bridge (t_1, t_2) of $G^*_{\bar{s}}$, (OA.7) is Step 5's tight bound; Lemma OA.14 also holds at widths $\bar{s} > \overline{\theta}$, where (ii) is not available, and there B falls and then rises. Let $\rho := \bar{s} - p$ and $c := t_2 - p$, so that $B = p + K/\rho + q(\rho - c)/\rho^2$ for $\rho \geq c$. Then

$$\frac{dB}{d\bar{s}} = \frac{\chi(\rho)}{\rho^3}, \qquad \chi(\rho) := \rho\, q'(\rho - c) - 2q(\rho - c) - K\rho, \qquad \chi''(\rho) = \rho\, q''' = \rho > 0.$$

So if $t_2 \leq \overline{\theta}$ and $\chi(\overline{\theta} - p) \leq 0$, the convex χ is nonpositive on an interval that starts at $\overline{\theta} - p$ and nonnegative beyond it, B is nonincreasing and then nondecreasing on

$[\overline{\theta}, \infty)$, and

$$\max_{\bar{s} \in [\overline{\theta}, S]} B(\bar{s}) = \max\{B(\overline{\theta}), B(S)\} \qquad \text{for every } S \geq \overline{\theta}. \tag{OA.8}$$

Proposition OA.10. *Let F satisfy (i) and (iii) of Theorem 3, with $2p - \underline{\theta} < \overline{\theta}$. Let $u^* := \max_{\bar{s} \in [2p-\underline{\theta}, \overline{\theta}]} u_F(\bar{s})$ and $s_T := p + \mathrm{Var}(F)/(u^* - p)$, and suppose that*

(P) $s_T \leq \overline{\theta}$, or there are $t_1 < t_2 \leq \overline{\theta}$ such that $\chi(\overline{\theta} - p) \leq 0$, $B(\overline{\theta}) \leq u^*$ and $B(s_T) \leq u^*$.

Then $u_F \leq u^*$ in every symmetric equilibrium of every rule. The uniformizing rule of a maximizing width attains u^*, and $u^* = \sup_{\bar{s} \geq 2p-\underline{\theta}} u_F(\bar{s})$.

Proof. The maximum exists, as $\mathrm{Var}(G^*_{\bar{s}})$ is continuous in $\bar{s}$, and $u^* \geq u_F(2p - \underline{\theta}) = \frac{4p-\underline{\theta}}{3} > p$ by (i), so s_T is well defined. Take a symmetric equilibrium. By Step 3, assume that Step 1's convex function is nondecreasing and $\mathrm{Cov}(s, V(s)) > 0$, as otherwise $u_F \leq p < u^*$, and let $\bar{s}$ and Y be as in Step 4. If $\bar{s} < \overline{\theta}$, Step 5 applies unchanged, as it uses only (i) for $\bar{s} \leq 2p - \underline{\theta}$ and only (iii) for $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$; it gives $u_F \leq u_F(2p - \underline{\theta})$ or $u_F \leq u_F(\bar{s})$, and both are at most u^*. Let $\bar{s} \geq \overline{\theta}$. Step 5's functions $w_1 = (x - p)^2$, $w_2 = 0$, $a(y) = y - p$ give $R = (s - y)^2 \geq 0$ without (ii), so $u_F \leq p + \mathrm{Var}(F)/(\bar{s} - p)$, which is at most u^* for $\bar{s} \geq s_T$. This covers every $\bar{s} \geq \overline{\theta}$ if $s_T \leq \overline{\theta}$. Otherwise, take $t_1 < t_2$ as in (P). For $\bar{s} \in [\overline{\theta}, s_T]$, as $\bar{s} \geq t_2$, Lemma OA.14 and (OA.8) give $u_F \leq B(\bar{s}) \leq \max\{B(\overline{\theta}), B(s_T)\} \leq u^*$. Finally, Proposition 12 gives an equilibrium with payoff $u_F(\bar{s})$ for every $\bar{s} \geq 2p - \underline{\theta}$, at a maximizing width in particular, so $u_F(\bar{s}) \leq u^*$ for all such $\bar{s}$. □

Any $t_1 < t_2 \leq \overline{\theta}$ will do in (P), so they may be rounded. A natural pair is the bridge of $G^*_{\overline{\theta}}$, if $G^*_{\overline{\theta}}$ has the shape of Proposition 12: then $B(\overline{\theta}) = u_F(\overline{\theta}) \leq u^*$ by the argument of Step 5.

Example OA.5. Let F be Beta$(\frac{1}{2}, \frac{3}{4})$, so $p = \frac{2}{5}$, $2p - \underline{\theta} = \frac{4}{5}$ and $\mathrm{Var}(F) = \frac{8}{75}$. This prior violates (ii) and satisfies (i), (iii) and (P), so by Proposition OA.10 its second best is that of the best uniformizing rule, $u^* \in [0.5755033, 0.5755059]$, i.e., 0.57550, at width $\bar{s}^* \approx 0.983$. The equilibrium $G^*_{\bar{s}^*}$ reveals the types below $t_1^* \approx 0.4097$, puts no mass on (t_1^*, t_2^*) with $t_2^* \approx 0.4654$, and pools the remaining types into signals spread uniformly over $[t_2^*, \bar{s}^*]$.

(i) and (iii) hold, and (ii) fails, for every Beta(a, b) with $a \leq b < 1$. Its density is U-shaped, with minimum at $m := \frac{1-a}{2-a-b}$, so (iii) holds by Lemma 19(ii), and F is concave on $[0, m]$ and convex on $[m, 1]$. Let $D(x) := F(x) - \frac{x}{2p}$ on $[0, 2p]$, where $2p \leq 1$. On $[0, \min(m, 2p)]$, D is concave, $D(0) = 0$, and $D > 0$ near 0, as $F(x)$ is of order x^a with

$a < 1$; so D is positive and then negative there. If $m < 2p$, then on $[m, 2p]$, D is convex with $D(2p) = F(2p) - 1 < 0$, so it is negative throughout if $D(m) < 0$ and changes sign once otherwise; if $m \geq 2p$, there is no convex part. On $(2p, 1)$, $F - 1 < 0$. So F minus the CDF of $U[0, 2p]$ changes sign once, from positive to negative, and as the means are equal, $U[0, 2p] \leq_{cx} F$, which is (i). As $b < 1$, $1 - F(x)$ is of order $(1 - x)^b$ near 1, while $1 - \bar{U}(x) = \frac{1-x}{2(1-p)}$, so $\int_x^1 (1 - F) > \int_x^1 (1 - \bar{U})$ for x near 1, and $F \leq_{cx} \bar{U}$ fails.

(P) holds. Take $t_1 = 0.527076406941$ and $t_2 = 0.568116732355$, the bridge of G_1^* rounded to twelve digits. Then $K = 0.0793460127$, $B(1) = 0.5748538578$ (which is $u_F(1)$), $\chi(0.6) = -0.0170128 < 0$, $s_T = 1.0077757$ and $B(s_T) = 0.5742656$. So $\max\{B(1), B(s_T)\} = 0.5748539$ falls short of the lower end of u^* by $6.5 \cdot 10^{-4}$. Here s_T is computed with that lower end, which is conservative, as it can only enlarge $[\overline{\theta}, s_T]$, and the bound $p + \mathrm{Var}(F)/(\bar{s} - p)$ for larger widths stays below the lower end of u^*.

How the numbers were checked. All computations use 40 significant digits, with F and its partial moments in closed form through regularized incomplete beta functions. The bridge of $G_{\bar{s}}^*$ solves its mass and mean equations; its root t_1 is located in a bracket on which the sign change was checked, and $u_F(\bar{s})$ is evaluated at both ends, keeping the smaller value. This gives the lower end of u^*, an attained payoff. For the upper end, $\mathrm{Var}(G_{\bar{s}}^*)$ is nondecreasing in $\bar{s}$, as $\bar{U}_{\bar{s}}$ increases in the convex order, so $u_F \leq p + \mathrm{Var}(G_{s_b}^*)/(s_a - p)$ on each cell $[s_a, s_b]$ of a grid of $[0.8, 1]$. An independent computation, with K by quadrature, reproduces all reported digits. The numbers are verified in high-precision floating point, not in interval arithmetic; every margin exceeds the numerical error by many orders of magnitude.

Condition (P) fails for near-symmetric U-shaped priors, for which the best width in $[2p - \underline{\theta}, \overline{\theta}]$ is $\overline{\theta}$ itself, and Beta$(\frac{1}{2}, \frac{1}{2})$ shows that (ii) cannot simply be dropped (see the footnote to Theorem 3).

OA.4. Extensions

This section relaxes the remaining assumptions. Sections OA.4.1 and OA.4.2 leave the well S-shaped class: the first studies its mirror image, and the second extends it and describes equilibria further away. Section OA.4.3 removes the rival.

OA.4.1 Reverse S-shaped selection rules

The well S-shaped class has a mirror image, in which the density of the selection rule is convex rather than concave. A concave density puts an agent's marginal gain from a

higher signal near a tie, whereas a convex density puts it at large margins.

Definition OA.1. A function $v : [-1, 1] \to [0, 1]$ is called *well reverse S-shaped* if $v(x) - \frac{1}{2}$ is odd and increasing, and $v'(x)$ is convex on $[-1, 1]$ and uniquely minimized at $x = 0$.

The following lemma transfers the equilibrium analysis to this class.

Lemma OA.15. *(i) For every well reverse S-shaped $v_2(x)$, there is a well S-shaped $v_1(x)$ with $\tilde{v}_2(x) = -\tilde{v}_1(x)$.*

(ii) Let $\tilde{v}_2(x) = -\tilde{v}_1(x)$. If $(G_1(s_1), G_2(s_2))$ is an equilibrium for $v_1(x)$ and priors (p_1, p_2), then $(\check{G}_1, \check{G}_2)$ is an equilibrium for $v_2(x)$ and priors $(1 - p_1, 1 - p_2)$, where $\check{G}_i$ is the distribution of $1 - s_i$ for $s_i \sim G_i$.

Proof. (i) Consider $v_1(x) = \frac{1}{2} + \frac{1}{2}x - a\tilde{v}_2(x)$. The convexity of $\tilde{v}_2'(x)$ and the uniqueness of its minimum at 0 are simply inverted into the concavity of $v_1'(x)$ and the uniqueness of its maximum at 0, and $v_1(x) - \frac{1}{2}$ is trivially odd and increasing for sufficiently small a, since $\tilde{v}_2'(x)$ is bounded. So $v_1(x)$ is well S-shaped for such a, while

$$\tilde{v}_1(x) = \frac{\frac{1}{2}x - a\tilde{v}_2(x) - x(\frac{1}{2} - a\tilde{v}_2'(0))}{|\frac{1}{2} - a - (\frac{1}{2} - a\tilde{v}_2'(0))|} = -\tilde{v}_2(x).$$

(ii) Since a normalization changes payoffs by a positive factor and terms independent of the signals, $v_k(x)$ and $\tilde{v}_k(x)$ have the same equilibria, $k \in \{1, 2\}$. The map $G \mapsto \check{G}$, where $\check{G}$ is the distribution of $1 - s$ for $s \sim G$, is a bijection between $\mathcal{G}(p)$ and $\mathcal{G}(1 - p)$. Since $\tilde{v}_1(x)$ is odd, $\tilde{v}_2\big((1 - s_i) - (1 - s_j)\big) = -\tilde{v}_1(s_j - s_i) = \tilde{v}_1(s_i - s_j)$, so each agent's payoff from $(\check{G}_1', \check{G}_2')$ under $\tilde{v}_2$ equals that from (G_1', G_2') under $\tilde{v}_1$ for all G_1', G_2'. Hence $(\check{G}_1, \check{G}_2)$ is an equilibrium for $\tilde{v}_2$ and priors $(1 - p_1, 1 - p_2)$. □

Hence, for equal priors, the unique equilibrium of a well reverse S-shaped $v(x)$ is $(\{b, 1\}; \{b, 1\})$, where b increases in p analogously to Theorem 1: each agent reveals good news rather than conceding bad news, and full revelation is reached for $p \leq \frac{1}{\mathcal{E}_{\tilde{v}}(1)}$, mirroring Proposition 1. In particular, such rules can deliver the first best for $p \leq \frac{1}{2}$: on $[0, 1]$, $v(x) = \frac{1}{2} + (\frac{1}{2} - \eta)x + \eta x|x|$ with $\eta \in (0, \frac{1}{2}]$ equals $\frac{1}{2} + \frac{1}{2}x - \eta x(1 - x)$, so it is revelation-accurate and strictly negatively balanced (Proposition 7).

The guarantee of Proposition 4, however, fails for many such rules.

Example OA.6. For $v(x) = \frac{1}{2} + \frac{1}{2}x^3$, the equilibrium is $(\{b, 1\}; \{b, 1\})$ with $b = \max(0, \frac{3p-1}{2})$, and $u_F(\frac{1}{3}) = \frac{5}{9} > 0.5458 = u_F(\frac{2}{5})$.

For these rules, informativeness often decreases in p unless the elasticity increases sharply, mirroring Lemma OA.3 and Corollary OA.1, and since $v(x)$ is convex on $[0,1]$, this effect can then outweigh the direct effect of the higher prior.

OA.4.2 Beyond the well S-shaped class

This section shows that the results of Section 3 hold, under a nondegeneracy condition, on a wider class of selection rules, that the bound of Theorem 2 does too, but that of Proposition 9 does not, and that further away equilibrium signals may have non-binary support.

OA.4.2.1 A wider class

Let $v'(x)$ be absolutely continuous and non-increasing on $[0,1]$, and let $w(x) := -v''(x) \geq 0$ there. Consider the ratio condition

$$w(x)\,w(y) \leq w(x+\delta)\,w(y+\delta) \qquad \text{for all } x, y, \delta > 0 \text{ with } x+y+\delta \leq 1. \tag{RC}$$

Where $w > 0$, (RC) says that $\frac{w(c-x)}{w(x)}$ is non-increasing in x on $(0,c)$ for every $c \in (0,1]$. A concave $v'(x)$ has a non-decreasing $w(x)$, so every well S-shaped rule satisfies (RC), but (RC) also lets $w(x)$ fall. In fact, a positive $w(x)$ satisfies (RC) if and only if $w = w_0 \cdot k$ with w_0 log-concave and symmetric about $\frac{1}{2}$ and k non-decreasing; well S-shaped rules are those with a constant w_0. For the converse, let $\varphi := \ln w$ and, for $0 < u < u' \leq \frac{1}{2}$, let $e(u,u')$ be the infimum of $\sum_i \big(\varphi(x_i + t_i - t_{i-1}) - \varphi(x_i)\big)$ over partitions $u = t_0 < \cdots < t_n = u'$ and points $x_i \leq t_{i-1}$. By (RC), e is additive and non-negative, $\varphi(1-u') - \varphi(1-u) \leq e(u,u') \leq \varphi(u') - \varphi(u)$, and $e(u+c, u'+c) \leq e(u,u')$, so $\ln w_0(u) := -e(u, \frac{1}{2})$, extended symmetrically about $\frac{1}{2}$, is concave, and w/w_0 is non-decreasing. If $w(x)$ is log-concave, (RC) need only be checked at $c = 1$. For $\xi \sim N(0, \sigma^2)$, $w(x) \propto x e^{-x^2/2\sigma^2}$ is log-concave and $(\ln w)'(x) + (\ln w)'(1-x) = \frac{1}{x(1-x)} - \frac{1}{\sigma^2}$, so (RC) holds if and only if $\sigma \geq \frac{1}{2}$, a quarter of the variance that well S-shapedness requires ($\sigma \geq 1$). For logistic ξ, (RC) holds if and only if its scale is at least $\frac{1}{2\ln(2+\sqrt{3})} \approx 0.38$, half the threshold for well S-shapedness.

Lemma OA.16. *Under* (RC), $\overline{v}(s_i)$ *is S-shaped against every* G_j. *Conversely, if* $\overline{v}(s_i)$ *is S-shaped against every* $\{0, c\}$, *then* (RC) *holds almost everywhere.*

Proof. Almost everywhere, $\overline{v}''(s) = R(s) - L(s)$ with $R(s) := \int_{(s,1]} w(t-s)\,dG_j(t)$ and $L(s) := \int_{[0,s)} w(s-u)\,dG_j(u)$. For $u < s < s' < t$, (RC) with $x = s-u$, $y = t-s'$ and

$\delta = s'-s$ gives $w(t-s')\,w(s-u) \le w(t-s)\,w(s'-u)$, and integrating over $t > s'$ and $u < s$ gives $R(s')L(s) \le R(s)L(s')$. So $\overline{v}''(s) \le 0$ implies $\overline{v}''(s') \le 0$, and $\overline{v}'(s)$ is quasi-concave. For the converse, against $\{0, c\}$ with $\mathbb{P}(s_j = c) = q$, $\overline{v}''(s) = q\,w(c-s) - (1-q)\,w(s)$ on $(0, c)$, which changes sign at most once from positive to negative for every q only if $\frac{w(c-s)}{w(s)}$ is non-increasing. □

So concavity of $v'(x)$ is more than the analysis of Section 3 needs. Let $v'(x)$ also be uniquely maximized at 0, which under (RC) means $w > 0$ on $(0, 1)$, and let $m(a) := \frac{\int_0^a t w(t)dt}{\int_0^a w(t)dt}$ be the mean of w on $[0, a]$. Then $\mathcal{E}_{\tilde{v}}(a) = \frac{a}{a-m(a)}$, so $\overline{p} = m(1)$, and the pooled signal of Theorem 1 solves $p = m(a)$. By (RC), $w(a-t) \ge w(t)$ for $t < \frac{a}{2}$, so $m(a) \ge \frac{a}{2}$, which replaces Lemma 3: $\overline{p} \ge \frac{1}{2}$, and $a \le 2p$. Theorem 1 holds as stated. If, in addition, no ratio $\frac{w(c-x)}{w(x)}$ is constant on an interval $(0, \varepsilon)$, then Lemma 1, Propositions 1–4 and Corollary 1 hold as stated. This holds when (RC) is strict, and for a well S-shaped rule it means that $v'(x)$ is not linear near 0. The proofs of Appendix A go through with the sign pattern of Lemma OA.16 in place of the concavity of $\overline{v}'(s_i)$, $m(a) \ge \frac{a}{2}$ in place of Lemma 3, and (RC) at $c = 1$ in place of the monotonicity of w in Step 2 of the proof of Proposition 2; in the proof of Proposition 3, $\partial_q N$ needs to be positive only at the root of N, where it is. Without this condition, but with $v'(x)$ not linear near 0, the equilibrium is still unique for all priors, except when w is symmetric about $\frac{1}{2}$ and $p_2 = \frac{1}{2} < p_1$: then agent 2 reveals fully, agent 1's value function against $\{0, 1\}$ is linear, and his equilibrium signals form a continuum.

The bound of Theorem 2 holds on the wider class. Since equilibria depend on $v(x)$ only through $w(x)$, take $v(1) = 1$ and $v'(1) = 0$, which maximizes the firm's utility given $w(x)$, and let $T(x) := x - \frac{x^2}{2}$, so that $v^*(x) = \frac{1}{2} + T(x)$ on $[0, 1]$. Lemma 6, $v(x) \le v^*(x)$, fails: if $w(x)$ is symmetric about $\frac{1}{2}$ and not constant, then $v(x) > v^*(x)$ on $(0, 1)$, and as w concentrates at $\frac{1}{2}$, $v(x)$ tends to the rule of $\xi \sim U[-\frac{1}{2}, \frac{1}{2}]$, which selects the higher signal for sure at every gap of at least $\frac{1}{2}$. But such a rule pools more: its $m(a)$ exceeds $\frac{a}{2}$. The two effects are tied by

$$v(a) - \tfrac{1}{2} \le T\big(\min\{2m(a), 1\}\big) \qquad \text{for all } a \in (0, 1],$$

which replaces Lemma 6 and, as $m(a) \ge \frac{a}{2}$, follows from it on the well S-shaped class. Its proof, which I only sketch, replaces k by a constant, which can only raise the left side and lower the right one; for $a \le \frac{1}{2}$, it then compares the symmetric log-concave w_0 with an exponential tent $e^{\gamma \min(t, 1-t)}$ with the same mass on $[0, a]$, and it reduces $a > \frac{1}{2}$

to $1-a$. With $p = m(a)$, the firm's utility is $p + 2p(1-\frac{p}{a})(v(a)-\frac{1}{2}) \leq p + p \cdot T(2p)$ for $p < \frac{1}{2}$, the bound of Theorem 2. So the extra accuracy is exactly offset by less revelation, and no rule satisfying (RC) beats $v^*(x)$: the gains of Subsection 4.2 require leaving this class.

For a general prior, the equilibrium part of Proposition 9 extends: Lemmas 9–11 hold, with the sign pattern of Lemma OA.16 in place of concavity and only minor changes otherwise, so (G_q, G_q) is an equilibrium for some $q \in (0, \frac{1}{2}]$, and the unique one under the condition above. The payoff bound does not extend, as with a general prior the extra accuracy need not cost revelation. Let $w(x) \propto x(1-x)$, i.e. $v(x) = \frac{1}{2} + x - x^3 + \frac{x^4}{2}$ on $[0,1]$, and let the types be 1 and $\frac{1}{2}$ with probabilities $\frac{3}{4}$ and $\frac{1}{4}$. Full revelation is an equilibrium under $v^*(x)$, as the atom at 1 exceeds $\frac{1}{2}$, and under $v(x)$, as it exceeds $\frac{5}{8}$. The only gap is $\frac{1}{2}$, at which $v(\frac{1}{2}) = \frac{29}{32} > \frac{28}{32} = v^*(\frac{1}{2})$, so the firm obtains $\frac{487}{512}$ under $v(x)$ and $\frac{484}{512}$ under $v^*(x)$. The same holds for $w(x) \propto \left(x(1-x) + \frac{1}{100}\right)(1 + \frac{x}{10})$, for which (RC) holds strictly, so that the equilibrium is unique.

OA.4.2.2 Non-binary support equilibrium

Outside the class defined by (RC), equilibrium signals need not have binary support, and their structure varies considerably with the selection rule.

Lemma OA.17. *Let $v(x)$ be real-analytic on $[-1,1]$ with $v''(x) > 0$ for all $x \in [-1,0)$. Then for all $p_i, p_j \in (0,1)$ and $G_j \in \mathcal{G}(p_j)$, every optimal signal for $(\overline{v}(s_i, G_j), p_i)$ has finite support.*

Proof. Since $v(x)$ is real-analytic on $[-1,1]$, it extends analytically to $\{z \in \mathbb{C} : dist(z,[-1,1]) < \delta\}$ for some $\delta > 0$, hence $\overline{v}(s_i) = \int_0^1 v(s_i - s_j) dG_j(s_j)$ extends analytically to $\{z \in \mathbb{C} : dist(z,[0,1]) < \delta\}$. Let $l(s_i)$ be a line supporting $\overline{v}^{cv}(s_i)$ at p_i. Then $l(s_i) \geq \overline{v}(s_i)$ on $[0,1]$, and every optimal signal is supported on $\{s_i \in [0,1] : \overline{v}(s_i) = l(s_i)\}$. If this set were infinite, it would have an accumulation point in $[0,1]$, and by the identity theorem $\overline{v}(s_i) = l(s_i)$ on $[0,1]$, so $\overline{v}''(0) = 0$. However, $\overline{v}''(0) = \int_0^1 v''(-s_j) dG_j(s_j) > 0$, since $v''(-s_j) > 0$ for $s_j \in (0,1]$ and G_j puts positive mass on $(0,1]$ as $p_j > 0$. □

These conditions hold for the selection rules generated by i.i.d. Gumbel or normal shocks of any variance, so for these natural examples, equilibrium signals are discrete. By the identity theorem, an analytic $\overline{v}(s_i)$ coincides with a line either on a finite set or on all of $[0,1]$, and the latter is ruled out by the convexity of $v(x)$ on $[-1,0]$. Still, simulations[1] show that the number of support points grows as the noise decreases,

[1] For $p = 0.3$ and $\xi \sim N(0, \sigma^2)$, the equilibrium G_i^* has 5 support points for $\sigma = 0.1$ and 9 for $\sigma = 0.05$.

mirroring the classical result for games on the unit square with bell-shaped kernels (see Karlin (1959)), in which optimal strategies are finitely supported and the number of support points grows without bound as the kernel sharpens.

In contrast, a selection rule that is not analytic can induce an equilibrium signal with a continuous part, already for exponential shocks.

Example OA.7. Let $v(x)$ be generated by i.i.d. $\varepsilon_i \sim Exp(\lambda)$, i.e. the Laplace CDF $v(x) = \frac{1}{2} + \frac{1}{2} \cdot \text{sign}(x) \left(1 - e^{-\lambda|x|}\right)$. Then for all $p \leq \frac{1}{2}$, both agents playing

$$G = \frac{1}{2(1+\lambda p)} \left(\delta_{\{0\}} + \delta_{\{2p\}}\right) + \frac{\lambda p}{1+\lambda p} U[0, 2p]$$

is a symmetric equilibrium.

In general, such equilibria are hard to find explicitly. If an equilibrium signal of agent j has an interval $[a, b]$ in its support, $\overline{v}(s_j)$ must be linear on $[a, b]$, which is an integral equation for G_i^*:

$$\overline{v}'(s_j) = \int_0^1 v'(s_j - s_i) dG_i^*(s_i) = const \quad \forall s_j \in [a, b].$$

Closed-form solutions are known only for special kernels,[2] such as those with a rational Fourier transform, as in Example OA.7, or piecewise polynomial ones, which include the triangular selection rules of Appendix B.4.

OA.4.3 A single agent

Without a rival, the firm compares the agent to an outside option of known value $c \in (p, 1)$: she hires him if his score exceeds c, with probability $a(s) := \mathbb{P}(s+\varepsilon \geq c)$, where ε has a continuous CDF F with $F(0) = \frac{1}{2}$. Her gain over not hiring is $W := \mathbb{E}_G[a(s)(s-c)]$, and her first best is $p(1-c)$. The agent maximizes $\mathbb{E}_G[a(s)]$ over $G \in \mathcal{G}(p)$.

Proposition OA.11. *Let $c \in (p, 1)$.*

(i) Without noise, $W = 0$. With noise, $W \leq p \min(\frac{1}{2}, 1-c)$.

(ii) If $\varepsilon \sim U[-u, u]$ with $u < c$ and $c + u \leq 1$, the agent's unique best reply is $\{0, c+u\}$, and $W = \frac{pu}{c+u}$. In particular, if $c > \frac{1}{2}$ and $u = 1 - c$, he reveals fully, and the firm attains her first best.

[2] This is a Fredholm equation of the first kind with a difference kernel on a finite interval; see Sakhnovich (2015) for the general theory and Polyanin and Manzhirov (2008) for solvable kernels.

(iii) The firm can attain her first best only if $c \geq \frac{1}{2}$.

Proof. (i) Without noise, $a(s) = \mathbb{1}_{s \geq c}$ lies below $\frac{s}{c}$ on $[0,1]$, touching it only at 0 and c, so the agent's unique best reply is $\{0, c\}$, and $W = \frac{p}{c}(c - c) = 0$. With noise, the split $\{0, c\}$ is feasible since $p < c$, and it gives the agent $(1 - \frac{p}{c})a(0) + \frac{p}{c}a(c) \geq \frac{p}{2c}$, as $a(c) = 1 - F(0) = \frac{1}{2}$ by continuity. Hence $\mathbb{E}_G[a(s)] \geq \frac{p}{2c}$ in equilibrium, and since $a \leq 1$ and $s \geq 0$,

$$W = \mathbb{E}_G[a(s)s] - c\,\mathbb{E}_G[a(s)] \leq p - c \cdot \frac{p}{2c} = \frac{p}{2}.$$

Also $W \leq \mathbb{E}_G[(s - c)^+] \leq p(1 - c)$, since $(s - c)^+ \leq s(1 - c)$ on $[0,1]$.

(ii) Here $a(s) = 0$ on $[0, c-u]$, $a(s) = \frac{s-c+u}{2u}$ on $[c-u, c+u]$, and $a(s) = 1$ on $[c+u, 1]$. Let $l(s) := \frac{s}{c+u}$. Then $l > a$ on $(0, c-u]$, $l - a$ is affine on $[c-u, c+u]$, positive at $c-u$ and zero at $c+u$, and $l > a$ on $(c+u, 1]$. So l supports the concavification of a, touching a only at 0 and $c+u$, and by Kamenica and Gentzkow (2011, Corollary 2), his unique best reply is $\{0, c+u\}$, with $\mathbb{P}(s = c+u) = \frac{p}{c+u}$. As $a(0) = 0$ and $a(c+u) = 1$, $W = \frac{p}{c+u}(c+u-c) = \frac{pu}{c+u}$. If $c > \frac{1}{2}$ and $u = 1-c$, then $u < c$, $c+u = 1$, and $W = p(1-c)$.

(iii) Since $a \leq 1$ and $(s - c)^+ \leq s(1 - c)$ with equality only at $s \in \{0, 1\}$, $W = p(1-c)$ requires $G = \{0, 1\}$, $a(0) = 0$ and $a(1) = 1$. Full revelation is a best reply only if a lies below its chord from $(0,0)$ to $(1,1)$, i.e., $a(s) \leq s$ on $[0,1]$, and at $s = c$ this gives $\frac{1}{2} \leq c$. □

Without noise, the agent splits the prior into $\{0, c\}$, and the firm hires him exactly when she is indifferent, so she gains nothing from the signal. Noise devalues a narrow lead over c, as it does a narrow lead over a rival in the main model, so the agent must prove more. The bound in (i) holds as the agent can always split into $\{0, c\}$, winning with probability at least $\frac{p}{c}a(c) = \frac{p}{2c}$: at the outside option, noise makes hiring a coin flip. The firm thus hires with probability at least $\frac{p}{2c}$, each hire costs c, and she can hire at most the mass p of good types. By (ii), uniform noise of half-width $\min(c, 1-c)$ attains this bound, strictly for $c > \frac{1}{2}$, in the limit for $c \leq \frac{1}{2}$: it is the widest noise that never hires a revealed bad agent and never rejects a revealed good one. As after Proposition 1, a tie is a coin flip, which is why full revelation requires $c \geq \frac{1}{2}$.[3] A privately known threshold can likewise force a single sender to reveal fully (Kolotilin et al., 2017); here the threshold is random to both sides, and the noise is unbiased.

[3]The bound uses only $a(c) \geq \frac{1}{2}$, so it holds for any noise with median 0. With mean-zero noise instead, rare shocks far beyond the posterior scale can move the mean without affecting decisions, and the firm can approach her first best for every c.

More generally, if the density of ε is unimodal with its peak at 0 and is not constant on $[0, c]$, then $a(s)$ is convex on $[0, c]$ and concave on $[c, 1]$, and the agent splits the prior into $\{0, y\}$ with $y > c$. The firm then gains if and only if p exceeds

$$\hat{p} := \frac{y\,c\,(1 - F(c))}{a(y)(y - c) + c\,(1 - F(c))}$$

Here $1 - F(c)$ is the probability that noise hires a revealed bad agent, so $\hat{p} = 0$ if $\varepsilon \leq c$, and for normal noise, $\hat{p}$ is close to 0 for small variances and rises with it. Two differences from the main model stand out. Without a rival, the noiseless firm learns nothing useful, whereas with two agents, she already learns much (Boleslavsky and Cotton, 2015). And a single agent's threshold is fixed, so a unimodal density suffices for bad news signal, whereas a rival's randomization makes the threshold random, and the well S-shaped class requires a concave density. The comparison also shows the role of commitment: a firm facing a single agent that could commit to any rule would hire only on a conclusive signal and attain her first best without noise. A firm comparing two agents must sometimes select the lower signal, but with binary types she still attains her first best, for instance with a pass/fail rule (Proposition 8); with a continuum of types and a non-convex prior, she cannot, and her best rule blurs the signals (Theorem 4).

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