Competitive Persuasion with a Noisy Judge

Igor Karpov*

November 17, 2026

Click here for the most recent version

Abstract

Can a firm choosing between two agents gain from judging them with error? I show that she can when the agents design the information she receives. Noiseless judging makes even a narrow lead decisive, so agents conceal bad news not to lose; noise makes a narrow lead worth little, so revealing pays more. For a broad class of noise and binary types, the equilibrium is unique: each agent sometimes admits being bad and otherwise pools, and if agents are likely enough to be good, both reveal fully, delivering the first best, unlike under noiseless judging. Moderate noise is best: it curbs concealment without reversing the ranking of posteriors, and the right amount beats noiseless judging at every prior. With a continuum of types, the case for noise survives: under regularity conditions, the firm prefers to judge with error unless the prior density is nondecreasing, and if it is nonincreasing, her best rule tosses a coin only among low posterior means.

Keywords: Bayesian persuasion, multiple senders, competition in persuasion, contests, noisy evaluation

JEL Classification: C72, D82, D83

*University of Chicago. I am very grateful to Doron Ravid, Ben Brooks and Emir Kamenica for their guidance and support. I also thank Alex Frankel, Joe Root, Phil Reny and the student workshop audience at the University of Chicago for useful comments and discussion. All errors are my own.

1. Introduction

A capital committee must fund one of two projects this quarter. Each sponsor decides how rigorous a pilot to run before the meeting, knowing that the committee's verdict will turn partly on things that have nothing to do with the projects: who presents second, how long the meeting has run. The committee would like to see past all of that. This paper asks whether it should want to.

Such imperfect verdicts, biased or merely arbitrary, are well recognized. In Scharfstein and Stein (2000), divisional rent-seeking leads to "socialist" internal capital markets, in which weaker divisions are subsidized by stronger ones. In Prendergast and Topel (1996), evaluators who lack objective performance measures act on personal preferences, favoring some subordinates beyond their true performance. And Pier et al. (2018) find essentially no agreement among experienced reviewers rating the same NIH grant applications, all of which were eventually funded: ratings of the same application were no more similar than ratings of different ones. The usual response is to engineer such imperfections away, through more reviewers, more structured evaluations, and more objective criteria. A few research funders have nonetheless moved the other way: they now fund by lot among proposals that clear a quality bar, citing reviewer bias, reviewer burden, and the hope of bolder proposals (Fang and Casadevall, 2016; Heyard et al., 2022; Röbbecke and Simon, 2023). Only the hope of bolder proposals concerns how applicants respond, and none concerns what the competitors choose to reveal about themselves, the channel I study.

The question is whether a decision-maker should eliminate all noise from her evaluation when the competitors know that it is noisy and adapt their reports to it. It arises whenever three features come together. First, a single prize is at stake, with no price attached: the decision-maker cannot pay more for the better competitor, so all she controls is whom she selects. Second, the competitors decide how much information

to reveal about themselves after learning how the decision-maker will choose between them. Third, the decision-maker can make the verdict more or less accurate, so the competitor with the higher posterior need not always win. I call the decision-maker *the firm*, a noisy judge when her verdict carries error, and the competitors *agents*. In the baseline, each agent is either a good or a bad type; Section 5 allows a continuum of types. Each agent simultaneously commits, at no cost, to a signal, that is, to a distribution of his posterior expected type; the firm observes the resulting posterior with a random error and selects the agent with the higher score. For many standard noise distributions and equal priors, this is also what she would do on seeing only the scores, so committing to the rule costs her nothing there.

My key observation is that noise of a suitable shape weakens each agent's incentive to conceal. An agent's chance of winning depends only on how far he looks ahead of his rival, in the same way for both agents; I call this dependence the firm's selection rule, and every such rule is generated by some noise. A noiseless firm always selects the agent who looks better, so her rule is a step: a narrow lead is as good as a wide one, and an agent revealed as bad loses unless his rival is too. Each agent therefore spreads his posteriors to land just above his rival's, blending his bad type with his good one: Boleslavsky and Cotton (2015, Lemma 3) show that both agents reveal only partially, randomizing over a continuum of posteriors. The firm then obtains less than 80% of her first best whenever agents are good with probability below one third.

With noise, an agent's chance of winning instead rises smoothly with his lead. A narrow lead is worth little, conceding bad news no longer means certain defeat, and the shape of the noise decides whether an agent gains more from gambling on being revealed as good or from pooling, and so how much information reaches the firm. The price is accuracy: where the chance of winning is near one half, selection is close to a coin flip, so the firm more often passes over the agent who looks better.

An example. Suppose that each agent is equally likely to be good and bad. In the unique equilibrium of the noiseless rule, each agent's posterior is uniformly distributed on $[0, 1]$, so the firm sometimes selects a bad agent over a good one and obtains about 89% of her first best. Now suppose the firm scores each agent by his posterior plus an independent draw from $U[0, 1]$. Against a rival who reveals his type, an agent's chance of winning is linear in his posterior, so revealing is a best reply; against any other signal it is not, and, since the agents' game is constant-sum, full revelation by both agents is the unique equilibrium. As a good type always outscores a bad one, noise only decides between agents of the same type, so the firm attains her first best.

Main results. The example relies on uniform noise and an even prior; the results below show how far this extends. The analysis moves from positive to normative: Section 3 takes a tractable class of noisy rules and asks what agents reveal under it, Section 4 lets the firm choose within such rules and then, as a benchmark, among all rules, and Section 5 does both with a continuum of types, where the benchmark becomes the main object. Section 3 characterizes the equilibrium when the difference between the two evaluation errors has a concave density with a single peak, which I call a *well S-shaped* rule; rules generated by i.i.d. normal or Gumbel shocks qualify if the shocks' variance is large enough. Under such a rule, whatever his rival reveals, an agent gains from gambling at low posteriors and from playing safe at high ones, so he uses a *bad news* signal: he sometimes reveals the bad type and otherwise pools, with full revelation and pooling as the edge cases. The equilibrium is unique whenever the priors are equal, and for every pair of priors unless the density is linear next to its peak. With equal priors, a single statistic governs it, the elasticity of the normalized rule at the pooled posterior: the firm sees the bad posterior with probability equal to its inverse, and both agents reveal fully once the prior is high enough. With unequal priors, each agent reveals more when his rival is better, and a much better favorite

reveals nothing while the underdog reveals fully. Under every well S-shaped rule, the firm gains from symmetrically better agents, even if they reveal less.

Section 4 turns to the design of the selection rule with equal priors. Within the well S-shaped class, a more elastic rule suppresses information, and for i.i.d. normal or Gumbel shocks, more noise means more elastic. The firm's best rule in the class is therefore at its edge: $v^*(x)$, generated by i.i.d. uniform shocks, induces the most information and selects most accurately. Scaling these shocks, the firm does best at the least noise that keeps signals binary, which depends on the prior: more noise only blurs selection, and less pushes the equilibrium toward the noiseless one. As Figure 1 shows, the best amount of uniform noise beats the noiseless rule at every prior, $v^*(x)$ does so for $p > 0.21$, and both attain the first best once $p \geq \frac{1}{2}$; below $\frac{1}{2}$, rules built for likely bad agents attain it, but no i.i.d. shocks generate them.

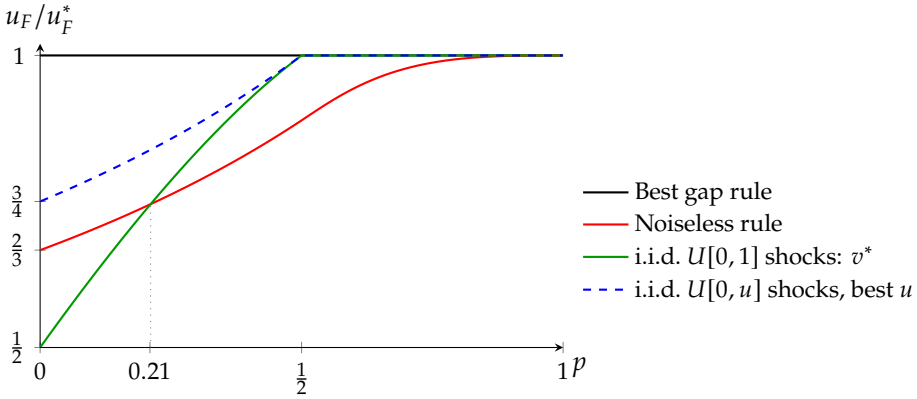


Figure 1: The firm's share of the first best under equal priors p, by selection rule; for uniform shocks narrower than $U[0,1]$, in the firm-preferred equilibrium

Subsection 4.3 asks which *gap rules*, as I call rules that depend only on the gap between the posteriors, attain the first best, which with binary types requires full revelation and a rule that always selects a revealed good agent over a revealed bad one. If the rule must deliver full revelation for priors both below and above $\frac{1}{2}$, only the linear rule does, as low priors rule out any rise of the rule above the line through

its endpoints and high priors any fall below it; it leaves agents indifferent among all signals, so robustness costs uniqueness. If the firm knows on which side of one half the prior lies, a rich family of rules makes full revelation the unique equilibrium, but for likely bad agents no i.i.d. shocks generate one. These limits are specific to gap rules: a designer who measures posteriors perfectly and can commit to any rule attains the first best for every pair of priors in the unique equilibrium, with rules responding to the posteriors' levels, from a pass/fail rule, which tosses a coin between an agent who is very likely good and one very likely bad, to rules arbitrarily close to the linear one.

Section 5 relaxes the binary prior, the most dispersed prior with a given mean. Under a well S-shaped rule, each agent still reveals his types below a threshold and pools the rest, never revealing more than the lower half, and $v^*(x)$, which makes him reveal exactly that half, remains the best rule in the class. The comparison with the noiseless rule turns on the shape of the prior. Under a convex prior, the noiseless rule attains the first best, which essentially no other rule can. Under regularity conditions, every other prior admits a rule that beats it, so even a designer with perfect measurement and full commitment should blur the posterior means: noise moves a little winning probability from the top of each pool to its bottom, so agents stay indifferent while spreading their pooled posterior means more widely. For concave priors and their close relatives, such as many single-peaked priors, the best rule makes the agents' scores uniform, as in the example, and generically strictly beats the noiseless rule and every fully revealing rule; for concave priors, it is a random pass mark, which compares agents exactly at the top and tosses a coin only among low posterior means.

Online Appendix OA.4 collects extensions: under reverse S-shaped rules, agents reveal good news; beyond the well S-shaped class, the results for binary types, including the optimality of $v^*(x)$, hold under a weaker condition on the noise; with a single agent, noise works the same way, by devaluing a narrow lead; and with more than two

agents, the linear rule, read as a uniform bonus to one randomly drawn agent, makes full revelation strictly dominant (Online Appendix OA.5).

1.1 Literature review

This paper contributes to the Bayesian persuasion literature (Kamenica and Gentzkow, 2011; Kamenica, 2019), in particular to its strand on competing senders. Most results of Gentzkow and Kamenica (2017) require a Blackwell-connected information environment, which fails when each sender can only reveal information about himself, as here, and their comparison with collusion is vacuous, since the agents' game is constant-sum. Each agent's best reply solves a single-sender problem with an S-shaped value function, so it is an upper censorship, as characterized by Kolotilin et al. (2022, Theorem 1) for smooth value functions; restated in quantiles, the result extends to every well S-shaped rule and an arbitrary prior (Lemma 9). With a general prior, I verify best replies through a convex function lying above the value function (Dworczak and Martini, 2019, Theorem 1), and restrict selection rules to posterior means, as is standard when the receiver's payoff is linear in the state (Gentzkow and Kamenica, 2016; Kolotilin, 2018). To a single agent, the rule acts like a receiver with a privately known threshold (Kolotilin et al., 2017), here the rival's posterior plus noise, and for a single sender, making a bell-shaped threshold density more diffuse leads to more information (Curello and Sinander, 2025, Theorem 2 and Section 5.2). Here, the threshold's distribution is an equilibrium object, set by the rival's signal.

The closest paper is Boleslavsky and Cotton (2015), part of the literature on grading policies (Ostrovsky and Schwarz, 2010; Chan et al., 2007; Popov and Bernhardt, 2013): two schools design grades, and an evaluator hires the graduate with the higher posterior. I add evaluation noise to this game, and its shape decides how much agents reveal. Under the noiseless rule, the equilibrium is unique, posteriors are uniform on an

interval, possibly with atoms at 0 and 1, and schools never reveal fully. Informativeness increases with the number of senders, both in this setting, where it becomes full in the limit (Au and Kawai, 2020, Theorem 2), and when senders also compete in prices (Hwang et al., 2026); the latter also characterize the noiseless equilibrium with a continuous prior; with costly inspection, the same convexity condition governs full revelation (Ding et al., 2023). Banerjee et al. (2026) bound the first-best loss of noiseless competition; a noisy judge can recover much of it, sometimes all. In search markets, whether competition raises informativeness depends on what sellers observe and how buyers search (Board and Lu, 2018); here it depends on the firm's selection rule.

An imperfect evaluation can also benefit the evaluator by strengthening investment incentives, as strategic grading does in Boleslavsky and Cotton (2015) and a noisy signal of universities' relative quality does in Boleslavsky and Li (2026). Here the gain comes through information instead: a suitably noisy evaluation makes signals more informative. In this, noise plays a role familiar from other settings: noise in prices preserves the incentive to acquire information that fully revealing prices would destroy (Grossman and Stiglitz, 1980), and noise in a cheap-talk channel makes the receiver less responsive to each message, which curbs the sender's exaggeration and improves communication (Blume et al., 2007). Closer in spirit, an organization that can fund only one of two proposals induces more informative signals than an unconstrained one and is better off unless both are ex-ante strong (Boleslavsky and Cotton, 2018), and moderate attention costs can make a rationally inattentive consumer select the better of two disclosed products more often than when attention is free (Jain and Whitmeyer, 2026). In both, the gain comes from a constraint on the receiver; I instead treat the evaluation error as a primitive of the selection rule, whose shape determines the equilibrium and how much information the firm extracts.

My normative results relate to the idea that observing an agent more closely can

hurt the principal once the agent responds to it (Prat, 2005), and that a receiver can extract more information by committing to use it less: random costs of approval make a single sender indifferent among signals and elicit full revelation, at the price of burned utility (Tsakas et al., 2021), and when agents manipulate their data at a cost, so do a less responsive allocation rule, a test that passes some noncompliant agents, and a score that underweights some features (Frankel and Kartik, 2022; Perez-Richet and Skreta, 2022; Ball, 2025). Extra noise in an already noisy channel can benefit the sender and sometimes both players (Tsakas and Tsakas, 2021), and coarse grades can inform the public better by inducing participation (Harbaugh and Rasmusen, 2018). Here signals are costless and flexible, and noise works only through competition, by changing the curvature of each agent's value function; it costs the firm nothing when it decides only between agents of the same type. Closest to the firm's commitment, Gao and Luo (2026) let a designer commit to a mechanism mapping each sender's experiment and posterior into a random action: robust mechanisms depend only on posteriors and induce full revelation, and with competing senders the efficient allocation is robustly implementable. Bergemann et al. (2023) show that a receiver uncertain about the sender's preferences optimally commits to a quota. Here the firm commits only to a comparison of the posteriors, which, for gap rules, makes robustness costly (Subsection 4.3) and, with a single agent and an absolute standard, caps her payoff. A single receiver who could commit to any decision rule would need no noise; a firm comparing two agents must sometimes select the lower posterior; with binary types she still attains the first best, as a pass/fail rule shows (Proposition 8), but with a continuum of types and a non-convex prior, even her best rule blurs the posterior means (Subsection 5.2). In the game of Boleslavsky and Cotton (2018), Muratov (2025) lets the receiver choose how to break ties: asymmetric tie-breaking helps her and sometimes attains the first best; here the rule responds to every gap between the posteriors, and

with a continuum of types even the best rule randomizes away from ties. Randomized choice deters gaming (Ederer et al., 2018) and keeps improper reasons out of selection (Fang and Casadevall, 2016); unlike random matching, which saves costly signaling by weakening it (Hoppe et al., 2009), noise here makes costless signals more informative.

The agents' game is also a contest in which each agent chooses a distribution with a fixed mean, and noise gives it a difference-form success function (Lazear and Rosen, 1981; Hirshleifer, 1989; Che and Gale, 2000), the counterpart of the ratio form of Tullock (1980), with more noise playing the role of lower decisiveness. When contestants choose the riskiness of their performance, limiting competition helps a principal select the better one (Hvide and Kristiansen, 2003; Fang and Noe, 2022), and a noisy selection rule is a new instrument with the same effect. In effort contests, a noisy contest can also be optimal (Morgan et al., 2022; Letina et al., 2023), and the shape of the noise matters for effort (Drugov and Ryvkin, 2020), as it matters here for information.

2. Model

Agents. There are two agents and a firm (she). Each agent's type θ_i is drawn independently from $\{0,1\}$, with *prior* $p_i := \mathbb{P}(\theta_i = 1)$, written p if $p_1 = p_2$.

Information and Timing. Agents play a simultaneous-move game in which, before his type is realized, each agent i commits to a signal structure; once θ_i is realized, the firm observes a draw from it. As the prior is binary, a posterior is pinned down by its mean, so I identify each realization with its posterior mean $s_i := \mathbb{E}[\theta_i|s_i]$, and agent i chooses a distribution G_i of posterior means (a *signal*); I call s_i agent i's *posterior* (*posterior mean* with a general prior). Feasible signals for a prior p_i form $\mathcal{G}(p_i) := \{G_i : \text{supp}(G_i) \subseteq [0,1], \int_0^1 s_i \, dG_i(s_i) = p_i\}$ (Gentzkow and Kamenica, 2016, Remark 1 and Proposition 1). Section 5 allows general priors.

Selection. Agents choose signal structures knowing the firm's *selection rule*: a function $v : [-1, 1] \to [0, 1]$ such that $v(x) - \frac{1}{2}$ is odd, increasing[1] and continuous. Given posteriors (s_1, s_2), agent i is selected with probability $v(s_i - s_j)$. As $v(x) - \frac{1}{2}$ is odd, $v(s_1 - s_2) + v(s_2 - s_1) = 1$, so the firm always selects exactly one agent and treats the agents anonymously. The rule is exogenous until Section 4, where the firm chooses it.

Noisy evaluation. Such selection rules arise naturally when the firm judges the agents with noise. Suppose that, instead of observing s_i without noise, the firm observes agent i's *score* $s_i + \varepsilon_i$, where ε_i is a random shock, independent of s_i, s_j, reflecting the idiosyncrasies of whoever is judging – a particular committee member's mood, the order in which proposals were read, an unmodeled personal rapport.[2] Taking the scores at face value, the firm selects the agent with the higher score. Suppose that $\xi := \varepsilon_j - \varepsilon_i$ has a continuous distribution, symmetric around 0; symmetry holds, for instance, whenever the shocks are exchangeable. Then she selects agent i with probability

$$\mathbb{P}(s_i + \varepsilon_i > s_j + \varepsilon_j) = \mathbb{P}(\xi < s_i - s_j) = v(s_i - s_j),$$

where $v(x)$, the CDF of ξ restricted to $[-1, 1]$, is a selection rule. Conversely, every selection rule arises in this way, although i.i.d. shocks generate only some (Corollary 2). I usually give $v(x)$ explicitly, and otherwise say that it *is generated by* the shocks.

Example 1. If $v(x)$ is generated by ε_i i.i.d. Gumbel or normal, then $v(x)$ is the CDF of a logistic or normal distribution, respectively.

Payoffs. Each agent receives 1 if selected and 0 otherwise. If the firm selects agent i, she receives θ_i, so her expected payoff is the selected agent's posterior s_i. The shocks

[1] Throughout, increasing and decreasing are meant weakly; strict monotonicity is stated explicitly.

[2] The noise is additive on the scale of the firm's payoff; noise on another scale, such as log-odds, would give rules that depend on more than the gap between the posteriors.

that may generate the selection rule do not enter her payoff.

Commitment. The firm commits to her selection rule. The commitment need not be deliberate: some noise is unavoidable and cannot be filtered out by the firm, and everyone understands how it shapes the verdict. Nor is it a restriction for the leading noise distributions: if the shocks are i.i.d. with a log-concave density, such as normal, Gumbel or uniform, and both agents use the same signal, as in the equilibrium for equal priors, then by the monotone likelihood ratio property the firm's expected payoff $\mathbb{E}[s_i|s_i + \varepsilon_i]$ is non-decreasing in the score, so selecting the higher score is optimal even without commitment. For unequal priors, and for rules that i.i.d. shocks cannot generate, the commitment assumption can bind.

Equilibrium. As agent i commits before learning θ_i, he maximizes his ex-ante probability of being selected. An equilibrium of the game is a pair (G_1^*, G_2^*) such that

$$G_i^* \in \arg\max_{G_i \in \mathcal{G}(p_i)} \int_0^1 \int_0^1 v(s_i - s_j) dG_j^*(s_j) dG_i(s_i).$$

For any priors and rule, an equilibrium exists by Sion's minimax theorem: the game is constant-sum, as $v(x) - \frac{1}{2}$ is odd, payoffs are linear and continuous in G_i, and $\mathcal{G}(p_i)$ is convex and weakly compact. Equilibria are the pairs of max-minimizers of this constant-sum game, so they form a product set.

Interpretation. The model is deliberately abstract and says nothing about a price: agent i's type θ_i is simply the value the firm realizes from picking i, with no wage or other transfer attached to that choice. This fits any setting in which a decision-maker allocates one indivisible prize between two competing options and cannot pay more

to secure a better one. My running example is **internal capital budgeting**: a firm's capital committee (*the firm*) must fund exactly one of two project proposals this cycle, and each project's sponsor (*agent i*) controls how much hard information – the scope of a pilot, the rigor of a feasibility study – to place in front of the committee. The same primitives fit a **grant panel** choosing between two proposals, and **grading**, as in Boleslavsky and Cotton (2015): two schools choose grading policies, and an employer with one opening compares a graduate from each. School i fixes its grading policy, the signal, in advance, knowing only the share p_i of good students in its cohort, not whether any exact student will be good; the noise is the idiosyncratic impression an interview leaves. I use *the firm* and *agent* as generic labels.

Under any of these readings, the signal $G_i(s_i)$ is the design of the hard information about the project, applicant or student – the scope and pre-registration of a pilot study, the preliminary results in a grant proposal, a school's grading policy – summarized by the distribution of the posterior s_i that the agent is good. Equivalently, as in Boleslavsky and Cotton (2015), each type generates a more primitive random report – a grade, a pilot outcome, a preliminary result – whose distribution satisfies the monotone likelihood ratio property, so that, by Bayes' rule, the posterior increases in the report. I read θ_i as fit with what the firm needs, which an agent can hardly assess; what he controls, before any result exists, is how informative the test will be. If he knew more than the firm, this would be a signaling problem, which I do not study.

The shocks are pure choice imperfections: they move the firm's decision but are unrelated to the substance of the signals and to her payoff. An agent may also have outcome-relevant characteristics unrelated to his signal; I set these aside, since agents care only about the selection rule, and comparing the firm's utility across rules is cleaner when the shock is pure evaluation noise.

3. Equilibrium under Well S-Shaped Rules

This section is positive: taking the rule as given, I study the agents' game under the additional assumption that the selection rule $v(x)$ has a concave density with a unique mode. Subsection 3.1 shows that this restricts every best reply to a bad news signal. Subsection 3.2 characterizes the equilibrium: for equal priors, it is unique and governed by a single statistic of the rule. Subsection 3.3 shows that a more elastic rule suppresses information, and Subsection 3.4 that the firm gains from better agents.

3.1 Best replies

Definition 1. A function $v : [-1, 1] \to [0, 1]$ is called *well S-shaped* if $v(x) - \frac{1}{2}$ is odd and increasing, and $v'(x)$ is concave on $[-1, 1]$ and uniquely maximized at $x = 0$.

Since $v(x) - \frac{1}{2}$ is odd, $v'(x)$ is even, so a concave $v'(x)$ is always maximized at $x = 0$. Requiring this maximum to be unique only rules out densities that are flat around 0, including the linear selection rule. Without this requirement, the equilibrium may fail to be unique: if $v(x)$ is linear on some $[-b, b]$, then for priors (p, p) with $p \leq b$, both $(\{p\}; \{p\})$ and $(\{0, b\}; \{0, b\})$ are equilibria: against either signal, an agent's winning probability is concave in his own posterior on $[0, 1]$ and linear on $[0, b]$, so every signal supported on $[0, b]$ is a best reply. If $v(x)$ is generated by $\varepsilon_i, \varepsilon_j$, well S-shapedness means that ξ has a concave density on $[-1, 1]$ with a unique mode.

Example 2. For $v(x)$ generated by i.i.d. Gumbel or normal shocks, there is a threshold $\overline{V}$ such that $v(x)$ is well S-shaped if $\mathrm{Var}[\varepsilon_i] \geq \overline{V}$. So is the rule generated by i.i.d. $\varepsilon_i \sim U[0, u]$ with $u \geq 1$, although its triangular density is linear on $[0, 1]$.

For equal priors, the results below need only well S-shapedness, so they cover rules whose density is linear next to its mode, such as the uniform rules of Example 2; for $u = 1$, this is the firm's best rule (Theorem 2). For unequal or non-binary priors,

uniqueness also requires that $v'(x)$ is *not linear near* 0, i.e., not linear on any $[0, \varepsilon]$, which makes every best reply unique (Lemma 1).

Let $\overline{v}(s_i, G_j) := \int_0^1 v(s_i - s_j) dG_j(s_j)$ be the *value function* of agent i, written $\overline{v}(s_i)$ from here on for simplicity when G_j is fixed. By choosing a signal G_i, agent i solves a Bayesian persuasion problem as in Kamenica and Gentzkow (2011):

$$\max_{G_i \in \mathcal{G}(p_i)} \int_0^1 \overline{v}(s_i, G_j) dG_i(s_i) \tag{BP}$$

A signal G_i that solves (BP) is *an optimal signal* for $(\overline{v}(s_i), p_i)$, and *the optimal signal* if no other signal does. To find one, I use the *concavification* $\overline{v}^{cv}(s_i)$ of $\overline{v}(s_i)$, its smallest concave majorant. A signal G_i is optimal for $\overline{v}(s_i)$ if and only if it is optimal for $\overline{v}^{cv}(s_i)$ and supported on the *concavification support* $\text{supp}(\overline{v}^{cv}) := \{s_i : \overline{v}(s_i) = \overline{v}^{cv}(s_i)\}$. Optimality for the concave $\overline{v}^{cv}(s_i)$ is easy to check (Kamenica and Gentzkow, 2011, Corollary 2): if $\overline{v}^{cv}(s_i)$ is linear on an interval around p_i, every feasible signal supported on that interval is optimal; otherwise, only pooling at p_i is.

A well S-shaped $v(x)$ helps because if $v'(x)$ is concave on $[-1, 1]$, then $\overline{v}'(s_i)$ is concave on $[0, 1]$ for every G_j, which gives $\overline{v}(s_i)$ the following shape: $\overline{v}(s)$ is *S-shaped* on $[a, b]$ if it is convex on $[a, c]$ and concave on $[c, b]$ for some $c \in [a, b]$. As $v(x)$ is continuously differentiable, so is $\overline{v}(s_i)$, and then $\overline{v}(s_i)$ is S-shaped if and only if $\overline{v}'(s_i)$ is quasi-concave. Since $\overline{v}'(s_i)$ is in fact concave, $\overline{v}(s_i)$ is S-shaped for every well S-shaped $v(x)$ and every G_j. Online Appendix OA.4.2 gives a weaker condition on $v''(x)$, necessary and sufficient for $\overline{v}(s_i)$ to be S-shaped against every G_j, under which the results of this section hold, with an analogue of the non-linearity of $v'(x)$ near 0.

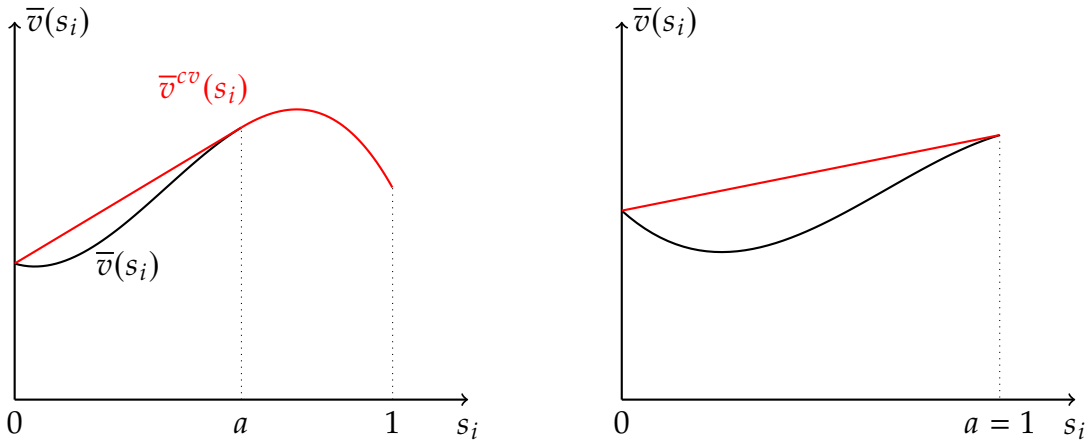
So $\overline{v}(s_i)$ has at most two regions of different convexity, and its concavification consists of at most two parts, which restricts the optimal signals (Lemma 1(ii)). Throughout, a signal with at most two points is identified with its support.

Lemma 1. *If $v(x)$ is well S-shaped, then*

(i) $\overline{v}(s_i, G_j)$ is S-shaped on $[0,1]$ for all $p_j \in (0,1)$ and $G_j \in \mathcal{G}(p_j)$.

(ii) An optimal signal for $(\overline{v}(s_i), p_i)$ is $\{0, a\}$ if $p_i < a$ and $\{p_i\}$ otherwise, where a is the largest $s_i \in [0,1]$ at which the secant of $\overline{v}$ from 0 to s_i is steepest, the secant at $s_i = 0$ being the tangent. If $v'(x)$ is not linear near 0, it is the optimal signal.

Figure 2: Illustration of Lemma 1 (ii)



(a) $\text{supp}(G_i) = \{0, a\}$ or $\{p_i\}$ (b) $\text{supp}(G_i) = \{0, 1\}$

This is not the image of $\overline{v}(s_i)$, but of $\overline{v}(s_i) - bs_i$ for some $b > 0$ for clarity. Since $cav(f(x) + bx) = cav(f(x)) + bx$ for all x, b, this does not influence the optimal signal.

Part (i) was shown above. For (ii), the concavification of an S-shaped $\overline{v}(s_i)$ coincides with the secant from $\overline{v}(0)$ to $\overline{v}(a)$ on $[0, a]$ and with $\overline{v}(s_i)$ on $[a, 1]$. Hence $\{0, a\}$ is an optimal signal if $a > p_i$, and $\{p_i\}$ if $a \leq p_i$, and if $v'(x)$ is not linear near 0, these are *the* optimal signals. Figure 2 illustrates both cases.

Lemma 1 thus restricts an agent's optimal signal to a narrow class, the *bad news* signals: the agent reveals the low type with some probability and otherwise induces a pooled posterior, with full revelation ($\{0,1\}$) and pooling ($\{p_i\}$) as the edge cases. Binary support is not special: two-point signals are the pure strategies of an agent,

every other signal is their mixture, and against any continuous value function some signal with at most two support points is a best reply. What a well S-shaped $v(x)$ adds is that, unless pooling is optimal, there exists such a best reply that has 0 in its support, and that it is unique if $v'(x)$ is not linear near 0. The point a depends only on the rival's signal, and the agent's own prior only determines whether he pools at p_i or plays $\{0, a\}$. The concavity of the density is what makes agents concede bad news: under the reverse S-shaped rules of Online Appendix OA.4.1, they reveal good news instead.

By the Bayesian constraint, positive scaling of $v(x)$ and adding a linear function to it do not change the concavification support, and hence do not change equilibria for any (p_1, p_2). I represent each such equivalence class of selection rules by the normalization

$$\tilde{v}(x) := \frac{v(x) - \frac{1}{2} - x \cdot v'(0)}{|v(1) - \frac{1}{2} - v'(0)|}.$$

It satisfies $\tilde{v}(0) = \tilde{v}'(0) = 0$ and $\tilde{v}(1) = -1$, which simplifies the exposition, and it is well defined for every well S-shaped $v(x)$: as $v'(x)$ is uniquely maximized at $x = 0$, $v(1) - \frac{1}{2} = \int_0^1 v'(x)dx < v'(0)$.

The key characteristic of $\tilde{v}(x)$ is its elasticity $\mathcal{E}_{\tilde{v}}(x) = \frac{x\tilde{v}'(x)}{\tilde{v}(x)}$. Since $\tilde{v}(0) = 0$, $\mathcal{E}_{\tilde{v}}(a)$ is the ratio of the slopes of the tangent to $\tilde{v}(x)$ at $x = a$ and of the secant from 0 to a, so it measures how curved $\tilde{v}(x)$ is on $[0, a]$. This curvature shapes $\overline{v}(s_i)$, and through it the pooled posterior a of Lemma 1: the elasticity pins down when agents reveal fully (Proposition 1) and where they pool otherwise (Theorem 1).

3.2 Equilibrium

3.2.1 Full revelation

A natural first question is whether the firm can ever get the first best information, full revelation by both agents, in equilibrium. Under a well S-shaped $v(x)$, the answer is

simple: both priors must be high enough.

Definition 2. For a given well S-shaped $v(x)$, let $\overline{p} := 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)} = 1 - \frac{1}{|\tilde{v}'(1)|}$.

Proposition 1. *Let $v(x)$ be well S-shaped. Then both agents playing full revelation is an equilibrium if and only if $p_1, p_2 \geq \overline{p}$, and then it is the unique equilibrium unless $\tilde{v}(x) = -x^2$ on $[0,1]$ and $\max(p_1, p_2) > \frac{1}{2} = \min(p_1, p_2)$.*

Competing with a sufficiently good rival induces an agent to reveal fully. If agent j plays $\{0,1\}$, the normalized value function of agent i is

$$\overline{v}(s_i) = (1-p_j)\tilde{v}(s_i) + p_j\tilde{v}(s_i - 1) = (1-p_j)\tilde{v}(s_i) - p_j\tilde{v}(1-s_i).$$

By Lemma 1, it is S-shaped, so $\{0,1\}$ is optimal if and only if the secant from 0 to 1 is weakly flatter than the tangent at $x = 1$ (see Figure 2b):

$$\overline{v}(1) - \overline{v}(0) \leq \overline{v}'(1) \Leftrightarrow \tilde{v}(1) \leq (1-p_j)\tilde{v}'(1) \Leftrightarrow p_j \geq 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)}.$$

For uniqueness, as the game is constant-sum, all equilibria form a product set, so any signal played in equilibrium is a best reply to full revelation. An S-shaped $\overline{v}(s_i)$ can touch its secant from 0 to 1 at some $s \in (0,1)$ only by coinciding with it on $[s,1]$, which requires $p_j = \overline{p}$ and, by Appendix A.3, $\tilde{v}(x) = -x^2$ on $[0,1]$, so $p_j = \frac{1}{2}$; otherwise full revelation is the only best reply, and if both priors are $\frac{1}{2}$, Theorem 1 gives uniqueness.

Remark 1. For $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$, $\tilde{v}(x) = -x|x|$ and $\overline{p} = \frac{1}{2}$; for $(p_1, p_2) = (0.75, \frac{1}{2})$, agent 1's value function against $\{0,1\}$ is linear, and $\{0,1\}$ remains a best reply to any $G_1 \in \mathcal{G}(0.75)$, since $\mathbb{E}_{G_1}[s_1^2] \geq \frac{1}{2}$. So $(G_1; \{0,1\})$ is an equilibrium for every such G_1.

Proposition 1 contrasts with the noiseless benchmark[3] of Boleslavsky and Cotton (2015, Proposition 1), in which schools never use a fully informative grading policy.

[3]The noiseless rule $v(x) = \frac{1}{2} + \frac{1}{2} \cdot \text{sign}(x)$ is not a selection rule, as it is discontinuous at 0; Appendix B.1 gives its equilibria.

This is not a feature of well S-shaped rules alone: Section 4 shows that full revelation can be reached by selection rules far beyond this class.

Since $\mathcal{E}_{\tilde{v}}(1) \geq 2$ (Lemma 3), the threshold $\overline{p} = 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)}$ is at least $\frac{1}{2}$, so under a well S-shaped rule, full revelation requires both agents to be at least as likely good as bad.

3.2.2 Equal priors

For equal priors, Theorem 1 characterizes the equilibrium on the well S-shaped class. Unless it is full revelation, it is a bad news equilibrium in which the bad posterior $s_i = 0$ has probability equal to the inverse elasticity at the pooled posterior a.

Theorem 1. *Let $v(x)$ be well S-shaped and the priors (p, p). If $p \geq \overline{p}$, the unique equilibrium is $(\{0,1\};\{0,1\})$. If $p < \overline{p}$, the unique equilibrium is $(\{0,a\};\{0,a\})$, where a solves $p = a(1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)})$, and $\mathbb{P}(s_i = 0) = \frac{1}{\mathcal{E}_{\tilde{v}}(a)}$.*

The intuition is simple. Against $\{0, a\}$, an agent's normalized value function is

$$\overline{v}(s_i) = \mathbb{P}(s_i = 0)\tilde{v}(s_i) + \mathbb{P}(s_i = a)\tilde{v}(s_i - a) = \frac{a-p}{a}\tilde{v}(s_i) + \frac{p}{a}\tilde{v}(s_i - a),$$

where the second term $\tilde{v}(s_i - a) = -\tilde{v}(a - s_i)$ reflects $\tilde{v}(s_i)$ on $[0, a]$. For $\{0, a\}$ to be a best reply, the secant from $\overline{v}(0)$ to $\overline{v}(a)$ must be tangent at $s_i = a$. Since $\tilde{v}'(0) = 0$, the second term has zero derivative at a, so $\overline{v}'(a) = \frac{a-p}{a}\tilde{v}'(a)$, while the secant has the slope $\frac{\tilde{v}(a)}{a}$, the same for both terms. Hence, in equilibrium,

$$\frac{a-p}{a}\tilde{v}'(a) = \frac{\tilde{v}(a)}{a} \Leftrightarrow \frac{a-p}{a} = \frac{1}{\mathcal{E}_{\tilde{v}}(a)} \Leftrightarrow p = a\left(1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)}\right).$$

Such a is unique, continuous and strictly increasing in p: the pooled posterior rises with the prior. Pooling can be approached in the limit: for $\tilde{v}_n(x) = -x^n$ on $[0, 1]$, the theorem gives $a = \frac{np}{n-1} \to p$, and more generally $a \to p$ whenever the elasticity of

$\tilde{v}(x)$ diverges uniformly on $(0, 1)$ (Lemma OA.1). Once signals carry a variance cost, pooling is attained: agents pool for every prior up to a threshold that rises with the cost (Proposition OA.2(ii) in Online Appendix OA.2.1).

3.2.3 Unequal priors

For unequal priors, let $p_2 \leq p_1$, so that agent 1 is the favorite and agent 2 the underdog, and let $v'(x)$ be not linear near 0. By Lemma 1, every best reply is unique and takes one of three forms: pooling $\{p_i\}$, full revelation $\{0, 1\}$, or bad news $\{0, c\}$ with $p_i < c < 1$. In general, an agent's best reply depends on his rival's whole signal, but in the following example it depends only on the rival's prior, which makes the equilibrium explicit.

Example 3. Let $\tilde{v}(x) = -x^3$. Agent i's best reply is $\{0, \min(\frac{3p_j}{2}, 1)\}$, or $\{p_i\}$ if $p_i \geq \frac{3p_j}{2}$. So for $p_2 \leq p_1$, the unique equilibrium is $(\{0, 1\}; \{0, 1\})$ if $p_2 \geq \frac{2}{3}$, and otherwise

(i) $(\{p_1\}; \{0, 1\})$ if $p_1 \geq \max(\frac{3p_2}{2}, \frac{2}{3})$;

(ii) $(\{p_1\}; \{0, \frac{3p_1}{2}\})$ if $\frac{3p_2}{2} \leq p_1 < \frac{2}{3}$;

(iii) $(\{0, \frac{3p_2}{2}\}; \{0, 1\})$ if $\frac{2}{3} \leq p_1 < \frac{3p_2}{2}$;

(iv) $(\{0, \frac{3p_2}{2}\}; \{0, \frac{3p_1}{2}\})$ if $p_1 < \min(\frac{3p_2}{2}, \frac{2}{3})$.

The example shows what asymmetry does. Each agent would ideally pool at one and a half times his rival's prior. When the priors are close and low enough, both concede bad news, and each pooled posterior is set by the rival: the underdog, facing the better rival, pools higher and so reveals more. As the favorite's lead grows, the posterior at which he would pool falls below his own prior, so once $p_1 \geq \frac{3p_2}{2}$ he pools at his prior. The underdog, facing a rival who reveals nothing, gambles harder and reveals fully once $p_1 \geq \frac{2}{3}$. So an agent nearly sure to win plays safe, and one likely to lose gambles; under reverse S-shaped rules, the roles reverse.

Proposition 2. *Let $v(x)$ be well S-shaped with $v'(x)$ not linear near 0, and let $p_1, p_2 \in (0,1)$ with $p_2 \leq p_1$. Then the equilibrium is unique: it is $(\{0,1\};\{0,1\})$ if $p_2 \geq \overline{p}$, and otherwise it has one of the following four forms:*

(i) $(\{p_1\};\{0,1\})$*: the favorite pools, the underdog reveals fully;*

(ii) $(\{p_1\};\{0,b\})$ *with $p_2 < b < 1$: the favorite pools, the underdog concedes bad news;*

(iii) $(\{0,a\};\{0,1\})$ *with $p_1 < a$: the favorite concedes bad news, the underdog reveals fully;*

(iv) $(\{0,a\};\{0,b\})$ *with $p_1 < a \leq b < 1$, and $a < b$ if $p_1 > p_2$: both concede bad news.*

In Example 3, regions (i)–(iv) are exactly the cases listed there (Figure 3). For a general rule, the two thresholds depend on the rival's whole signal rather than on his prior, so the regions have curved boundaries. Every pair of forms except pooling by both agents is played for some priors; against pooling, an agent's value function is convex below the rival's prior and concave above it, so pooling is a best reply only for the agent with the strictly higher prior (proof of Lemma 5), and the underdog never pools at his prior. Without non-linearity of $v'(x)$ near 0, neither the uniqueness of the equilibrium nor these forms are guaranteed.

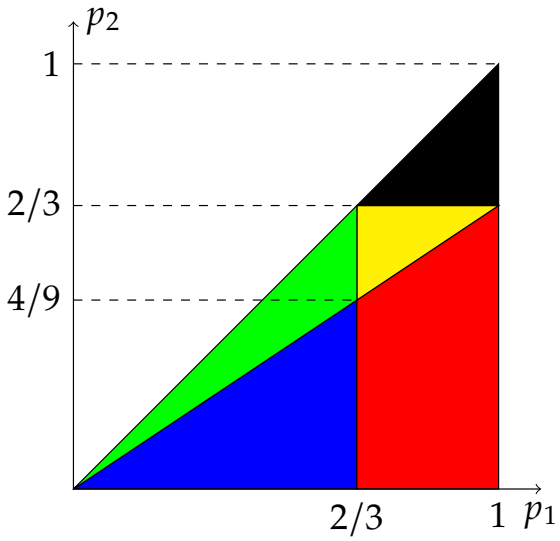
In regions (ii) and (iii) of Proposition 2, an agent's bad news signal depends only on his rival's prior. More generally, the rival forces an agent to reveal.

Proposition 3. *Let $v(x)$ be well S-shaped with $v'(x)$ not linear near 0. Then agent i's equilibrium high posterior is nondecreasing in p_j, strictly if he plays $\{0,a\}$ with $p_i < a < 1$.*

An agent's own prior usually has the opposite effect: in Example 3, his high posterior does not depend on p_i, so he reveals less as p_i increases. This effect need not be monotone, though.[4]

[4]Online Appendix OA.1.2 gives the region boundaries for a general rule, an example in which uniqueness and the forms of Proposition 2 fail when $v'(x)$ is linear near 0, an example in which an agent's signal is non-monotone in his own prior, and proofs of the unequal-priors claims in Subsection 3.4.

Figure 3: Equilibria for unequal priors in Example 3, $\tilde{v}(x) = -x^3$ on $[0,1]$



Colors are for the corresponding equilibrium type: black – $(\{0,1\};\{0,1\})$*; red* – $(\{p_1\};\{0,1\})$*; green* – $(\{0,a\};\{0,b\})$*; blue* – $(\{p_1\};\{0,b\})$*; yellow* – $(\{0,a\};\{0,1\})$.

3.3 Noise and informativeness

In the symmetric equilibrium of Theorem 1, a higher pooled posterior a means, for a fixed p, a more informative signal: the agent reveals his bad type more often, and revealing it less often is a garbling (Blackwell, 1953). So the selection rule affects information through a. If $\tilde{v}(x)$ has a constant elasticity, then, for a fixed p, a higher elasticity lowers $\mathbb{P}(s_i = 0)$ and hence a, making the signal less informative. The next corollary extends this argument to any two selection rules.

Corollary 1. *Let $v_e(x)$ and $v_n(x)$ be well S-shaped, let the priors be (p,p) with $p \in (0, \min(\overline{p}_e, \overline{p}_n))$, and let $(\{0,a_k\};\{0,a_k\})$ be the equilibrium for v_k, $k \in \{e,n\}$. If $\tilde{v}_e(x)$ is more elastic than $\tilde{v}_n(x)$, i.e. $\mathcal{E}_{\tilde{v}_e}(x) > \mathcal{E}_{\tilde{v}_n}(x)$ for all $x \in (0,1)$, then $a_e < a_n$.*

So a more elastic selection rule suppresses information, and within the well S-shaped class, the most informative signal is induced by the rules with $\mathcal{E}_{\tilde{v}}(x) = 2$, i.e. $\tilde{v}(x) = -x^2$ on $[0,1]$. For the natural examples, this reads intuitively in terms of noise.

Example 4. Numerically, for $v(x)$ from i.i.d. Gumbel or normal shocks, $\mathcal{E}_{\tilde{v}}(x)$ increases in $\mathrm{Var}[\varepsilon_i]$ on $[\overline{V}, +\infty)$ for every $x \in (0,1)$, so noisier shocks yield less information.

3.4 Better agents and the firm

It is natural to expect the firm to benefit from ex-ante better agents. But better agents need not reveal more: informativeness can fall in the prior, and this fall could outweigh the firm's direct gain from a higher prior.

For a well S-shaped $v(x)$ and priors (p, p), let (G, G) be the unique equilibrium and $u_F(p) := \mathbb{E}_G[s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1)]$ the firm's expected utility.

Proposition 4. *Let $v(x)$ be well S-shaped. Then $u_F(p)$ is strictly increasing in p.*

So, despite possibly decreasing informativeness, a firm that uses a well S-shaped rule is never harmed by ex-ante better agents. The result is not immediate: the equilibrium signals for different priors need not be ranked by first-order stochastic dominance or in the Blackwell order, and under a noisy selection rule the firm's payoff need not be monotone or convex in the posteriors. However, the change from p to $p' > p$ splits into two steps within the bad news class. First, keeping the prior p, the pooled posterior rises from $a(p)$ to $a(p')$: this spread reveals the bad type more often and widens the gap between the posteriors, so the firm selects the worse agent less often. Second, keeping the pooled posterior $a(p')$, the prior rises to p', which moves mass from 0 to $a(p')$, and on $\{0, a\}$ the firm's payoff is monotone. Both steps benefit the firm. This logic rests on the shape of $v(x)$: under a well S-shaped rule, the pooled posterior moves away from the revealed one as p increases. Under reverse S-shaped rules, it moves toward it, and $u_F(p)$ can fall: for $v(x) = \frac{1}{2} + \frac{1}{2}x^3$, the unique equilibrium gives $u_F(\frac{1}{3}) = \frac{5}{9} > 0.5458 = u_F(\frac{2}{5})$. Better agents need not make her mistakes rarer: the probability of selecting the agent with the lower posterior falls in p if $\mathcal{E}_{\tilde{v}}(x)$ is

non-decreasing, but can rise otherwise (Online Appendix OA.1.3 studies both this and how informativeness responds to the prior).

For unequal priors, this monotonicity extends to every region of Proposition 2 except region (iv), where both agents concede bad news: once the underdog stops revealing fully, his pooled posterior can fall as his prior rises, and the firm's utility with it. For the agents, a better own prior always helps and a better rival always hurts.

4. Designing the Selection Rule

This section turns from the positive analysis to the normative one: the firm's choice of the selection rule, first within tractable classes and then, as a benchmark, among all rules. Subsection 4.1 finds the best rule within the well S-shaped class, and Subsection 4.2 the optimal level of noise beyond it. Both depend on the prior, so Subsection 4.3 asks which rules implement full revelation robustly, and Subsection 4.4 compares gap rules with a benchmark designer who may use any rule.

4.1 The best well S-shaped rule

I now depart from an exogenous $v(x)$: for a fixed p, the firm chooses the selection rule to maximize $u_F(p)$, for instance, by choosing between committee structures, evaluation protocols or review panels. The natural examples show an important pattern: numerically, for $v(x)$ generated by i.i.d. Gumbel or normal shocks, $u_F(p)$ decreases in $\text{Var}[\varepsilon_i]$ on $[\overline{V}, +\infty)$ for every $p \in (0,1)$, so within each class the best rule lies at the "edge" of the well S-shaped class: the lowest level of noise that keeps $v(x)$ well S-shaped. Two questions arise. What is the best equilibrium utility within the well S-shaped class for each p, and which rules attain it? How does it compare with the noiseless rule? The answer to the second is not obvious. In the introduction's example, where $p = \frac{1}{2}$, the noiseless rule leaves each agent's posterior uniform on $[0,1]$, less

informative than the full revelation of the example's rule; there noise costs no accuracy, but in general it trades accuracy for information. The next theorem answers both.

Theorem 2. *Let $v(x)$ be well S-shaped. Then $u_F(p) \le u_F^{ws}(p) = p + \min(2p, 1) \cdot p(1-p)$, and the rule $v^*(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$ attains this bound for every $p \in (0, 1)$. Moreover, $v^*(x)$ gives the firm strictly more than the noiseless rule if and only if $p > \frac{3-\sqrt{3}}{6} \approx 0.21$.*

Since the first best utility is $u_F^*(p) := \mathbb{E}[\max(\theta_1, \theta_2)] = 2p - p^2 = p + p(1-p)$, $v^*(x)$ captures the share $\min(2p, 1)$ of the first best gain over random selection. No other well S-shaped rule attains the bound for $p < \frac{1}{2}$; for $p \ge \frac{1}{2}$, a rule attains it if and only if $p \ge \overline{p}$, so that agents reveal fully, and $v(1) = 1$ (Appendix B.2).

The rule $v^*(x)$ has a triangular density and is generated by two independent $\varepsilon_i \sim U[0, 1]$. Its optimality has a simple source. In the equilibrium $(\{0, a\}; \{0, a\})$, the firm's utility is $p + \mathbb{P}(s_1 \ne s_2) \cdot a \cdot \left(v(a) - \frac{1}{2}\right)$: she obtains p from a random pick and gains only when the posteriors differ, by their gap times her excess probability of selecting the higher one. This gain rises with the pooled posterior a and, at a given a, with the accuracy $v(a)$, and $v^*(x)$ is best on both counts. It induces the highest possible pooled posterior, $a = \min(2p, 1)$, and since a concave density on $[0, 1]$ declines no faster, in proportion, than the triangular density $1 - x$, and its mass there is at most $\frac{1}{2}$, $v^*(x)$ is the most accurate well S-shaped rule at every gap: $v(x) \le v^*(x)$ on $[0, 1]$. So $v^*(x)$ lies at the far edge of the class, and the firm trades accuracy for information only beyond it.

Figure 1 shows the share of the first best utility $u_F^*(p)$ attained by $v^*(x)$ and by the noiseless rule. The extra information $v^*(x)$ induces outweighs the cost of sometimes selecting a bad agent over a good one, most of all at high priors, where agents reveal fully and $v^*(x)$ attains the first best.

These comparisons rely on equal priors. If the underdog is more likely bad than good, a well S-shaped rule beats the noiseless rule only when the priors are close, and never when the favorite pools: this is proven for $v^*(x)$ and partly for other rules, and

numerical otherwise (Online Appendix OA.2.4). If both agents are more likely good than bad, $v^*(x)$ attains the first best; if only one is, another rule does (Subsection 4.3).

4.2 The optimal level of noise

Theorem 2 leaves open what happens beyond the edge of the well S-shaped class, where the noise is too small for the density of the shock difference to be concave. The selection rules generated by i.i.d. $\varepsilon_i \sim U[0, u]$, which have a triangular density on $[-u, u]$, cover both sides of this edge. For $u \geq 1$, $v(x)$ is well S-shaped with $\tilde{v}(x) = -x^2$ on $[0, 1]$, and $u = 1$ gives $v^*(x)$; for $u < 1$, $v'(x)$ is not concave on $[0, 1]$, and as $u \to 0$, $v(x)$ converges pointwise to the noiseless rule. For every u and p, a symmetric equilibrium can be constructed explicitly, and it has binary support if and only if $u \geq 2\min(p, 1-p)$; for smaller u, each agent randomizes over more than two posteriors. The construction, in Appendix B.4, rests on u-partitions: signals on an interval $[a, b]$ against which, under the triangular rule, an agent's value function is linear on $[a, b]$.

Proposition 5. *Let $v(x)$ be the CDF of a triangular distribution on $[-u, u]$, $u \in (0, 1]$, and let $u_F(p, u)$ be the firm's utility in the equilibrium (G_u, G_u) constructed in Appendix B.4.*

(i) If $p \leq \frac{1}{2}$, G_u is strictly increasing in u in the Blackwell order on $(0, 2p]$, and $G_u = \{0, 2p\}$ for $u \in [2p, 1]$. If $p > \frac{1}{2}$, $G_u = \{0, 1\}$ for $u \in [2 - 2p, 1]$.

(ii) If $p \leq \frac{1}{2}$, $u_F(p, u)$ is uniquely maximized at $u = 2p$ and strictly decreasing on $[2p, 1]$. If $p > \frac{1}{2}$, $u_F(p, u)$ equals the first best $u_F^*(p) = 2p - p^2$ if and only if $u \in [2 - 2p, 1]$.

(iii) As $u \to 0$, G_u converges weakly to the equilibrium signal for the noiseless rule.

For $u \geq 1$, $v(x)$ is well S-shaped with $\tilde{v}(x) = -x^2$ on $[0, 1]$, so the equilibrium signal does not change with u, while $v(x) = 1 - \frac{(u-x)^2}{2u^2}$ decreases in u on $(0, 1]$, and so does the firm's utility.

Part (i) shows that for $p \leq \frac{1}{2}$, less noise only suppresses information. For $p > \frac{1}{2}$ and $u < 2 - 2p$, this ranking breaks down. For instance, for $p = \frac{2}{3}$, G_u puts $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ on $\{0, \frac{1}{3}, 1\}$ for $u = \frac{1}{3}$ and $(\frac{4}{21}, \frac{4}{21}, \frac{13}{21})$ on $\{0, \frac{1}{4}, 1\}$ for $u = \frac{1}{2}$, and neither is more informative. The same holds for any two $u \in [1 - p, 2 - 2p)$ if $p \geq \frac{5}{9}$ (Appendix B.5). Still, on this range, the probability of revealing the good type and the variance increase in u.

Part (ii) shows that the firm's utility is maximized at the least noise at which agents play a binary signal, $\{0, 2p\}$ or $\{0, 1\}$; less noise breaks the binary structure and strictly lowers it. So the optimal noise is intermediate, trading off information from the agents against accuracy of selection. At least within this family, the firm never benefits from non-binary signals, so she may not need to go far beyond the well S-shaped class.

Part (ii) also compares the best rule in this family with the noiseless rule at every prior. For $p \leq \frac{1}{2}$, the noise $u = 2p$ gives $v(2p) = 1$, so the firm selects the higher of two different posteriors for sure and noise changes only how much agents reveal. Her utility is then $\frac{3p}{2}$, against $\frac{4p}{3}$ under the noiseless rule and $u_F^{ws}(p)$ under the best well S-shaped rule; as $p \to 0$, this is three quarters of the first best, against two thirds. For $p > \frac{1}{2}$, every $u \in [2 - 2p, 1]$ attains the first best. So the right amount of noise beats the noiseless rule at every prior, whereas $v^*(x)$ does so only for $p > 0.21$. Nor does any other noise distribution do better with binary signals: for every rule generated by i.i.d. shocks with a square-integrable density, every symmetric equilibrium $(\{0, a\}; \{0, a\})$ has $a \leq 2p$, so $u_F \leq \frac{3p}{2}$ (Online Appendix OA.4.4).

Part (iii) shows that there is no discontinuity at zero noise: as $u \to 0$, the equilibrium converges to the noiseless one of Boleslavsky and Cotton (2015). Simulations show the same convergence for logistic and normal noise.

Two caveats, however, point to robust design. First, the optimal noise is tied to the prior: for $p \leq \frac{1}{2}$, more noise than $u = 2p$ blurs selection and less breaks the binary signals, so a firm that misjudges p loses either way. Second, for $u < 1$ I do

not prove that the equilibrium is unique, so this comparison rests on the equilibrium of Proposition 13; in simulations of the discretized game, equilibria approach it. At the optimal noise $u = 2p$, however, no equilibrium gives the firm more than $\frac{3p}{2}$, and numerically the equilibrium is unique. A firm unsure of the priors thus needs a robust design, the subject of Subsection 4.3.

4.3 Attaining the first best

For $p < \frac{1}{2}$, no rule of Subsections 4.1–4.2 attains the first best. This subsection lets the firm choose any gap rule and asks which such rules attain it, first for a known prior and then robustly, for all priors in a given set.

I say that $v(x)$ is *revelation-accurate* if $v(1) = 1$, i.e. whenever agent i is revealed to be good and agent j bad ($s_i = 1$, $s_j = 0$), the firm selects agent i with probability 1. This is necessary for optimality: if $v(1) < 1$, an affine modification of $v(x)$ leads to the same equilibria and, unless both agents pool, gives the firm a strictly higher utility.

I also say that $v(x)$ *implements a fully revealing equilibrium (uniquely) for* (p_1, p_2) if, for these priors, full revelation by both agents is an (the unique) equilibrium. Together with revelation accuracy, this gives the firm $\mathbb{E}[\max(\theta_1, \theta_2)]$, the first best, for any priors; for equal priors, both conditions are also necessary.

Lemma 2. *For every selection rule $v(x)$ and priors (p, p), the firm's utility in every equilibrium is at most $u_F^*(p)$, and some equilibrium attains it if and only if $v(x)$ implements a fully revealing equilibrium for (p, p) and is revelation-accurate.*

The proof is simple. The firm's payoff $s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1)$ as a weighted average is not more than $\max(s_1, s_2)$. As $\max(s_1, s_2)$ is convex and piecewise linear, its expectation over $G_1, G_2 \in \mathcal{G}(p)$ attains $u_F^*(p)$ if and only if at least one agent reveals fully. With equal priors, the equilibria form a product set, so if full revelation is played in some equilibrium, so is $(\{0, 1\}; \{0, 1\})$. Hence the firm falls short of $u_F^*(p)$ in

every equilibrium unless $v(x)$ implements a fully revealing equilibrium, and in that equilibrium she attains it if and only if $v(x)$ is revelation-accurate. The result generally fails for $p_1 \neq p_2$: the set of equilibria is then not symmetric and may contain only equilibria in which exactly one agent reveals fully, as at the end of this subsection.

Against a fully revealing rival, an agent with posterior x wins with probability $\frac{1}{2} + (1-p)\hat{v}(x) - p\hat{v}(1-x)$, where $\hat{v}(x) := v(x) - \frac{1}{2}$, so $v(x)$ implements a fully revealing equilibrium for (p, p) if and only if this function lies below its secant from 0 to 1:

$$(1-p)\hat{v}(x) - p\hat{v}(1-x) \leq (x-p)\hat{v}(1) \quad \forall x \in (0,1).$$

For every $p \neq \frac{1}{2}$, a rich family of rules satisfies the condition, but the family has no tractable characterization for a single prior and is tuned to p. So suppose that the firm knows only that the priors lie in a set $S \subseteq (0,1)$. By Lemma 8 in Appendix B, $v(x)$ implements a fully revealing equilibrium for all $(p_1, p_2) \in S^2$ if and only if the condition above holds for every $p \in S$.

Proposition 6. *Let a selection rule $v(x)$ implement a fully revealing equilibrium for (p_l, p_l) and (p_h, p_h), where $p_l < \frac{1}{2} < p_h$. Then*

(i) $v(x)$ implements a fully revealing equilibrium for all $(p_1, p_2) \in (0,1)^2$;

(ii) $v(x) = \frac{1}{2} + ax$, where $a \in [0, \frac{1}{2}]$.

Adding the condition above at x and $1-x$ with weights $1-p$ and p gives $(1-2p)\big(\hat{v}(x) - x\hat{v}(1)\big) \leq 0$, so p_l rules out any upward deviation of $\hat{v}(x)$ from the line $x\hat{v}(1)$ and p_h any downward one. A linear rule satisfies the condition with equality for every p, and Lemma 8 extends this to all pairs of priors.

By Lemma 2, a firm that attains the first best at one equal prior below $\frac{1}{2}$ and one above must therefore use $v(x) = \frac{1}{2} + \frac{1}{2}x$. Under this rule every value function is linear,

so every pair of signals is an equilibrium, and the firm attains the first best only in her preferred one: full robustness leaves agents indifferent among all signals. With more than two agents, the same rule, a uniform bonus to one randomly drawn agent, makes full revelation strictly dominant, and no other bonus distribution attains the first best at every equal prior (Online Appendix OA.5).

The indifference disappears under slightly less robustness: suppose the firm knows whether both agents are more or less likely to be good than bad, i.e. $S = [\frac{1}{2}, 1)$ or $S = (0, \frac{1}{2}]$. The rules that implement full revelation on either set are characterized by the deviation g of $v(x)$ from its secant from 0 to 1.

Definition 3. Let $v(x)$ be a selection rule, $a := v(1) - \frac{1}{2}$, and $g(x) := v(x) - \frac{1}{2} - ax$. Then $v(x)$ is *balanced* if $g(x) = g(1-x)$ on $(0,1)$; *positively* [*negatively*] *balanced* if also $g(x) \geq 0$ [≤ 0] there; and *strictly* so if $g(x) > 0$ [< 0].

Like $\tilde{v}(x)$, $g(x)$ removes a linear part of $v(x)$, but it subtracts the secant from 0 to 1 rather than the tangent at 0. So equilibria depend on $v(x)$ only through $g(x)$, and revelation accuracy only through a; a well S-shaped $v(x)$ is positively balanced if and only if its normalization is $\tilde{v}(x) = -x^2$ on $[0,1]$.[5]

Proposition 7. *Let $v(x)$ be a selection rule.*

(i) $v(x)$ implements a fully revealing equilibrium for all $(p_1, p_2) \in [\frac{1}{2}, 1)^2$ [for all $(p_1, p_2) \in (0, \frac{1}{2}]^2$] if and only if it is positively [negatively] balanced.

(ii) If $v(x)$ is strictly positively [negatively] balanced, it implements a fully revealing equilibrium uniquely $\forall (p_1, p_2) \in (\frac{1}{2}, 1)^2$ $\left[\forall (p_1, p_2) \in (0, \frac{1}{2})^2\right]$.

So a rule that implements full revelation on a half-interval differs from a universal implementation $v(x) = \frac{1}{2} + ax$ of Proposition 6 only by a function g that is symmetric

[5] By Proposition 7(i), $v(x)$ is positively balanced if and only if it implements full revelation for all $p \in [\frac{1}{2}, 1)$, which by Proposition 1 means $\overline{p} \leq \frac{1}{2}$; by Lemma 3, $\overline{p} \geq \frac{1}{2}$, with equality if and only if its normalization is $\tilde{v}(x) = -x^2$ on $[0,1]$.

around $\frac{1}{2}$, non-negative for high priors and non-positive for low ones. Both properties come from the endpoints. As the conditions are linear in p, they hold on $[\frac{1}{2}, 1)$ $[(0, \frac{1}{2}]]$ if and only if they hold at $p = \frac{1}{2}$ and at $p = 1$ $[p = 0]$, and the latter gives $g \geq 0$ $[g \leq 0]$. At $p = \frac{1}{2}$, the value function $\frac{1}{2} + \frac{1}{2}\hat{v}(s_i) - \frac{1}{2}\hat{v}(1 - s_i)$ is odd around $s_i = \frac{1}{2}$, so it lies below its secant only if it coincides with it, i.e. only if $v(x)$ is balanced.

Part (ii) is sufficient for uniqueness: as equilibria form a product set, it suffices that $\{0, 1\}$ is each agent's unique best reply to itself, which fails at $p = \frac{1}{2}$, where any $v(x)$ that implements full revelation makes $\overline{v}(s_i)$ linear. It is not necessary, but being only a slight strengthening of (i), it leaves little room for other rules with a unique fully revealing equilibrium; for those, one must show that no signal supported on $\{s_i : g(s_i) = 0\}$ is a max-minimizer. If signals are costly, the linear rule collapses: any positive cost makes pooling the unique equilibrium, whereas for smooth costs, $v^*(x)$ keeps full revelation unless the prior is near one half (Proposition OA.3).

For high priors, evaluation noise suffices: the rule $v^*(x)$ of Theorem 2, generated by i.i.d. $U[0, 1]$ shocks, is indeed revelation-accurate and strictly positively balanced. For low priors, i.i.d. noise does not: the next result extends the bound $\overline{p} \geq \frac{1}{2}$ beyond the well S-shaped class, so no negatively balanced rule is generated by i.i.d. shocks.

Corollary 2. *If $v(x)$ is differentiable at 0 and 1 with $v'(0) > v'(1)$ and $\min(p_1, p_2) < \frac{1}{2}$, then $(\{0, 1\}; \{0, 1\})$ is not an equilibrium. In particular, this holds for every $v(x)$ generated by i.i.d. shocks with a square-integrable density.*

With unequal priors, the firm needs less, as the first best requires full revelation by only one agent. For instance, let $v(x) = \min\{1, \frac{1}{2} + x + \varepsilon x(1 - 2x)\}$ for $x \geq 0$, extended by $v(-x) = 1 - v(x)$, and let $\min(p_1, p_2) < \frac{1}{2} < \max(p_1, p_2)$. For $\varepsilon > 0$ small enough given the priors, the unique equilibrium attains the first best: the agent with the lower prior reveals fully, and the other plays $\{\frac{1}{2}, 1\}$. As the rule treats the agents alike, it does not need to know which agent is likely good. Any other rule that attains the first best

for such priors is linear or selects the higher posterior for sure from some gap on, so no rule generated by i.i.d. shocks as in Corollary 2 does (Appendix B.11).

4.4 A benchmark: general selection rules

Both limitations, indifference under full robustness and the failure of i.i.d. noise for low priors (Corollary 2), are specific to gap rules. To see how much they cost, consider a designer who may use any rule of the following kind.

Definition 4. A *general selection rule* is a measurable $H : [0,1]^2 \to [0,1]$ with $H(x,y) + H(y,x) = 1$ for all $x, y \in [0,1]$, so that $H(x,x) = \frac{1}{2}$: agent i is selected with probability $H(s_i, s_j)$. Gap rules are the case $H(x,y) = v(x-y)$.

Neither continuity nor monotonicity in the own posterior is required, and neither restriction would cost the firm anything. Every rule that attains a bound in this section or the next is nondecreasing in the own posterior, and the bounds hold among all rules: if a value function decreases somewhere on the support, pooling the lowest posteriors makes it nondecreasing and weakly raises the firm's payoff. Subsection 5.2 shows that continuity, too, costs the firm nothing in the limit.

Many such rules have no natural interpretation, so I treat them as a benchmark. It is demanding in two ways. The designer commits to any rule, however unnatural ex post, and she measures each agent's realization, the posterior mean of his type, perfectly: the rule is applied to s_i itself rather than to a noisy reading of it. The preceding subsections showed that eliminating all noise from the measurement can hurt the firm. The benchmark asks the converse question: how much could a firm gain if it could measure perfectly and commit to any use of the measurement? The answer never requires an unnatural rule, because the optimum can be sought within a simple class. With binary types, a deterministic pass/fail rule attains the first best.

Proposition 8. *Let types be binary and H be nondecreasing in the own posterior.*

(i) H attains the first best $2p - p^2$ in some equilibrium for all priors (p, p) with $p \in (0,1)$ if and only if $H(1,0) = 1$, $H(x,0) \leq \frac{1+x}{2}$ and $H(x,1) \leq \frac{x}{2}$ for all $x \in (0,1)$. If the last two inequalities are strict, then for all priors $(p_1, p_2) \in (0,1)^2$ the firm attains the first best in the unique equilibrium, which is fully revealing.

(ii) In particular, the pass/fail rule*, which selects an agent whose posterior is* 1 *over one whose posterior is below* 1 *and tosses a coin otherwise, attains the first best in the unique equilibrium for all $(p_1, p_2) \in (0,1)^2$.*

Part (i) parallels Proposition 6, where the analogous two inequalities force a gap rule to be linear. A general rule can satisfy both inequalities in (i) strictly: every rule $\frac{1}{2} + \frac{1}{2}\big(g(s_i) - g(s_j)\big)$ with g nondecreasing, $g(0) = 0$, $g(1) = 1$ and $g(x) < x$ on $(0,1)$ does, such as $g(x) = x^2$; the pass/fail rule is the case $g(x) = \mathbb{1}\{x = 1\}$. Each such rule is a lottery among the agents whose posteriors clear a bar drawn from g, as funders who draw lots among proposals above a quality bar do; a fixed bar below 1 fails, and a bar drawn uniformly gives the linear rule.

The pass/fail rule requires a strong commitment. Ex post, it ignores what the firm has learned: between an agent with posterior 0.95 and one with posterior 0.05, it tosses a coin, which is as noisy as selection can be. Ex ante, this is exactly what makes revelation pay: an interior posterior, however close to 1, buys nothing over revealing a bad type. Every rule of Proposition 8(i) departs from selecting the higher posterior in this way, as $H(x,0) \leq \frac{1+x}{2} < 1$, but most depart far less: with $g(x) = x - \varepsilon x(1-x)$, the rule selects the agent with posterior 0.95 over the one with 0.05 with probability 0.95 for every $\varepsilon \in (0,1)$, as the linear rule does. What the rules with strict inequalities share, and what the uniqueness needs, is that they respond to the levels of the posteriors, which no gap rule can do. An imperfect measure can still be just as good as a perfect one: the linear gap rule, at the boundary of both inequalities, attains the benchmark's first best too, in its firm-preferred equilibrium. With binary types, the benchmark is

thus trivial; Section 5 relaxes the binary prior and shows that with a concave prior the benchmark's own optimum deliberately blurs the measured posterior means.

5. Non-Binary Priors

With a general prior F, a signal G is feasible if and only if it is a mean-preserving contraction of F (Gentzkow and Kamenica, 2016, Remark 1 and Proposition 1), i.e. $\mathbb{E}_G[\phi] \leq \mathbb{E}_F[\phi]$ for every convex ϕ; the binary prior, the largest distribution on $[0,1]$ with mean p in the convex order, gives the richest set of feasible signals. I count the noiseless rule $v(x) = \frac{1}{2} + \frac{1}{2} \cdot \operatorname{sign}(x)$ as a gap rule and write $\underline{\theta} = \min \operatorname{supp}(F)$ and $\overline{\theta} = \max \operatorname{supp}(F)$ for the lowest and highest types.

With a general prior, two posteriors with the same mean can differ, for instance in how conclusive they are. Throughout, selection rules depend on the posteriors only through their means, the standard restriction when the receiver's payoff is linear in the state (Gentzkow and Kamenica, 2016; Kolotilin, 2018; Dworczak and Martini, 2019; Kolotilin et al., 2022), as the firm's is here. For the benchmark designer of Subsection 4.4 the restriction has bite: a rule that prefers degenerate posteriors to non-degenerate ones makes full revelation weakly dominant and attains the first best for every prior. Subsection 5.1 keeps gap rules, as in Section 3, and Subsection 5.2, the normative part, returns to the benchmark designer of Subsection 4.4.

5.1 Gap rules

Under a well S-shaped rule, an agent's value function is S-shaped against every signal of his rival, so, for a prior with a positive density, an upper censorship is a best reply (for smooth value functions, Kolotilin et al., 2022, Theorem 1): it reveals the types below a cutoff and pools all types above it into one posterior mean. I extend this result to arbitrary priors, with atoms and gaps, by defining censorship on quantiles:

the *quantile censorship* G_q reveals the lowest share q of types and pools the rest at their mean, splitting an atom at the cutoff if needed. Against every rival signal, some quantile censorship is a best reply (Lemma 9), so the equilibrium keeps the structure of Section 3: low types are revealed, and high types are pooled. Theorem 2 also extends to every prior: $v^*(x)$ is the best well S-shaped rule (Proposition 9). Accuracy and revelation still do not conflict: $v^*(x)$, the most accurate well S-shaped rule at every gap between the posterior means (Lemma 6), also induces the largest revealed share, $q = \frac{1}{2}$.

Proposition 9. *Let $v(x)$ be well S-shaped and F any prior. Then (G_q, G_q) is an equilibrium for some $q \in (0, \frac{1}{2}]$, and the unique equilibrium if $v'(x)$ is not linear near 0 or $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$. In every symmetric equilibrium in quantile censorships, $q \leq \frac{1}{2}$, and the firm's utility is at most its value in the median-censorship equilibrium of $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$.*

For any prior, the equilibrium is a quantile censorship. Either it solves a first-order condition, (QC) in Appendix C, or it reveals fully, pooling only types at $\overline{\theta}$, which requires an atom at $\overline{\theta}$ of mass at least $1 - q \geq \frac{1}{2}$ (Lemmas 10 and 11). For the binary prior, the agents reveal fully exactly when the atom at 1 has mass $p \geq \overline{p}$, where $\overline{p} \geq \frac{1}{2}$ (Theorem 1). For $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$, the best well S-shaped rule of Theorem 2, the first-order condition collapses: $q = \frac{1}{2}$ for every prior, so each agent reveals the types below the median and pools those above it, splitting an atom at the median so that exactly half the mass is revealed (Appendix C.4).

5.2 General selection rules with a general prior

From here on, the firm chooses any general selection rule H of Definition 4, as in Proposition 8; gap rules are a special case, and *rule* means a general selection rule throughout.[6] Against a rival who plays G, the agent's value function is $V_G(x) :=$

[6]Online Appendix OA.3 proves the supplementary claims of this subsection: the best fully revealing rule when $F \leq_{cx} \bar{U}$ fails, the uniqueness of the random-pass-mark equilibrium, the convergence of

$\int H(x, y)\, dG(y)$, his probability of being selected with posterior mean x.[7] The optimal rules below are discontinuous at ties, but smoothing them near ties gives continuous rules whose equilibrium payoffs converge to the same bounds.

In a symmetric profile (G, G), the firm's payoff is $u_F = p + 2\,\mathrm{Cov}_G\big(s, V_G(s)\big)$, so she cares about the covariance of an agent's posterior mean and his winning probability. Let V_F be the value function against a fully revealing rival, whose posterior mean is his type. A function V equals V_F on $\mathrm{supp}(F)$ for some rule if and only if $V(\theta) \leq_{cx} U[0, 1]$ for $\theta \sim F$ (Appendix C.8): noise only makes winning probabilities less dispersed than exact ranking, and the noiseless rule, with $V_F = F$ for atomless F, is the boundary case.

5.2.1 The first best

Proposition 10. *Let* $\mathrm{supp}(F) = [\underline{\theta}, \overline{\theta}]$. *Some general selection rule nondecreasing in the own posterior mean attains the first best in some equilibrium if and only if F is convex on $[\underline{\theta}, \overline{\theta})$ with no atom below $\overline{\theta}$. Then the noiseless rule attains it, with a unique, fully revealing equilibrium.*

A noisy gap rule selects the higher of two close posterior means with probability near one half, so it misranks fully revealing agents whose types are close, while the noiseless rule attains the first best, so the result of Theorem 2 reverses. More generally, the first best requires always selecting the higher type, so every equilibrium that attains it is fully revealing, and a rule attains it only if it coincides with the noiseless rule for $F \otimes F$-almost every pair of posterior means, and no other gap rule attains it (Appendix C.8.4). For priors with a positive density, Hwang et al. (2026, Lemma 1, Theorem 1 and Corollary 1) show that full revelation is an equilibrium of the noiseless rule if and only if F is convex, and that it is then the unique symmetric equilibrium.

continuous approximations of the optimal rules, and the regularity of the noiseless equilibrium for the remaining Beta priors and for priors with real-analytic densities.

[7] Without continuity an equilibrium need not exist; when equilibria exist, they are interchangeable and give each agent $\frac{1}{2}$, and I evaluate a rule at its firm-preferred equilibrium.

5.2.2 The best fully revealing rule

Define the affine value function

$$L(x) := \frac{1}{2} + \frac{x - p}{2(\overline{\theta} - p)}, \qquad \text{so } L(p) = \tfrac{1}{2},\ L(\overline{\theta}) = 1,$$

and let $\bar{U} := U[2p - \overline{\theta}, \overline{\theta}]$, the uniform distribution with the prior's mean and top.

Proposition 11. *Let F be any prior.*

(i) For every general selection rule with upper semicontinuous V_F under which full revelation is an equilibrium,

$$u_F \leq p + \frac{\mathrm{Var}_F(\theta)}{\overline{\theta} - p},$$

with equality if and only if $V_F = L$ F-almost surely, which is feasible if and only if $F \leq_{cx} \bar{U}$.

(ii) In every symmetric equilibrium (G, G) of every general selection rule with $\bar{s}_G := \max \mathrm{supp}(G) > p$, $u_F \leq p + \mathrm{Var}_G(s)/(\bar{s}_G - p)$. If $\bar{s}_G = \overline{\theta}$, this is at most the bound in (i), as $\mathrm{Var}_G(s) \leq \mathrm{Var}_F(\theta)$ for $G \leq_{cx} F$.

So the best fully revealing rule is *the narrowest uniformizing noise*. When agents reveal fully, it makes their scores uniform on $[2p - \overline{\theta}, \overline{\theta}]$, the narrowest interval centred at the mean that contains the prior's top. Such a rule exists when this uniform is a mean-preserving spread of the prior, i.e., when $F \leq_{cx} \bar{U}$. This is the logic of the example in the introduction, where uniform shocks make the scores of fully revealing agents uniform. Every concave prior satisfies $F \leq_{cx} \bar{U}$; it fails for non-uniform convex priors, for which the noiseless rule attains the first best anyway.[8]

When $F \leq_{cx} \bar{U}$ fails, the bound is not attained. For a single-peaked prior, $F \leq_{cx} \bar{U}$ if and only if $p \leq \frac{\underline{\theta}+\overline{\theta}}{2}$ (Lemma 19); otherwise, the best fully revealing rule ranks weak

[8] A concave prior has a nonincreasing density, possibly unbounded at $\underline{\theta}$, and possibly an atom at $\underline{\theta}$.

types exactly and applies the narrowest uniformizing noise only above a cutoff. When $F \leq_{cx} \bar{U}$, the best fully revealing rule attains the bound in (i). By (ii), another rule does better only if its equilibrium pools the top types into posterior means below $\overline{\theta}$. Pooling the top has two effects on the bound in (ii): it lowers the highest posterior mean $\bar{s}_G$, which raises the bound, but also lowers the signal dispersion $\mathrm{Var}_G(s)$, which lowers it. For Beta$(2,2)$, $p = \frac{1}{2}$, the first effect dominates: the best fully revealing rule attains the bound, 0.600, but the noiseless rule, whose equilibrium pools the top, gives 0.626.

The rule attaining the bound can be chosen nondecreasing in the own posterior mean. Every feasible nondecreasing value function, L included, arises from a lottery over coarse gradings: the firm draws a partition of the posterior means into intervals, selects the agent in the higher interval, and otherwise tosses a coin (Appendix C.8).

5.2.3 The second best

Under the best fully revealing rule, the agents' scores are uniform on $[2p - \overline{\theta}, \overline{\theta}]$. Targeting scores uniform on $[2p - \bar{s}, \bar{s}]$ instead, for any $\bar{s} \geq 2p - \underline{\theta}$, gives a family of rules indexed by the width $\bar{s}$. Let $\bar{U}_{\bar{s}} := U[2p - \bar{s}, \bar{s}]$ and $L_{\bar{s}}(x) := \frac{1}{2} + \frac{x-p}{2(\bar{s}-p)}$, its CDF on its support. For $\bar{s} < \overline{\theta}$, the uniform $\bar{U}_{\bar{s}}$ stops below the prior's top, so it is not a mean-preserving spread of F, and the agents must pool some top types. Let $G^*_{\bar{s}}$ be the most dispersed mean-preserving contraction of both F and $\bar{U}_{\bar{s}}$ (Appendix C.8). The *uniformizing rule of width* $\bar{s}$ is the rule whose noise turns a posterior mean drawn from $G^*_{\bar{s}}$ into a score drawn from $\bar{U}_{\bar{s}}$, so a posterior mean x beats a rival drawn from $G^*_{\bar{s}}$ with probability $L_{\bar{s}}(x)$, which rises linearly from $\frac{1}{2}$ at the mean to 1 at $\bar{s}$.

Proposition 12. *For every $\bar{s} \geq 2p - \underline{\theta}$, the uniformizing rule of width $\bar{s}$ has an equilibrium $(G^*_{\bar{s}}, G^*_{\bar{s}})$ with $u_F(\bar{s}) = p + \mathrm{Var}(G^*_{\bar{s}})/(\bar{s} - p)$. If $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$ and the set $\{x \in [\underline{\theta}, \bar{s}] : F(x) > L_{\bar{s}}(x)\}$ is an interval, as it is for concave F, then $G^*_{\bar{s}}$ reveals the types below some t_1, puts no mass on some (t_1, t_2), and on $[t_2, \bar{s}]$ has the density of $\bar{U}_{\bar{s}}$, i.e., pools the remaining types into*

uniformly spread posterior means.

Corollary 3. *Let F be concave on its support $[\underline{\theta}, \overline{\theta}]$. Then (G, G) with $G = U[\underline{\theta}, 2p - \underline{\theta}]$ is the unique equilibrium of the noiseless rule, and the firm's utility is $u_F = \frac{4p-\theta}{3}$.*

For concave priors, the family runs between two familiar ends. At width $2p - \underline{\theta}$, the uniformizing rule adds no noise and is the noiseless rule. By Corollary 3, its equilibrium signal is $U[\underline{\theta}, 2p - \underline{\theta}]$. For priors with a density, this symmetric equilibrium is also in Hwang et al. (2026, Corollary 1). At width $\overline{\theta}$, the equilibrium is fully revealing and gives the best fully revealing rule of Proposition 11. In between, agents reveal their low types and pool their high types into uniformly spread posterior means. The equilibrium has the shape of the censorship equilibrium of Proposition 9, with one difference: above the cutoff, a well S-shaped rule makes the value function concave, and a uniformizing rule makes it affine, so the posterior means of the high types are spread uniformly instead of being pooled at a single common posterior mean.

Under the noiseless rule, the firm's payoff for priors with $U[\underline{\theta}, 2p - \underline{\theta}] \leq_{cx} F$ depends on the prior only through its mean and its lower support point, so it does not grow with dispersion, while the first best does. So either end of the family can win, depending on how dispersed the prior is, as the following example shows.

Example 5. Let $p = \frac{1}{3}$ and $\text{supp}(F) = [0, 1]$, and compare $F(x) = 2x - x^2$ with its mean-preserving spread $F(x) = \frac{1+2x}{3}$, an atom of $\frac{1}{3}$ at 0 plus a uniform density. Their variances are $\frac{1}{18}$ and $\frac{1}{9}$. The noiseless rule gives $\frac{4}{9} \approx 0.444$ under both; the best fully revealing rule gives 0.417 and 0.500; and $v^*(x)$ of Theorem 2 gives 0.399 and 0.457.

For concave priors, and more generally for those of the next result, the best member of the family does at least as well as every rule, and generically neither end is optimal.

Theorem 3. *Let the prior F satisfy*

(i) $U[\underline{\theta}, 2p - \underline{\theta}] \leq_{cx} F$;

(ii) $F \leq_{cx} \bar{U}$;

(iii) for every $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, *the set* $\{x \in [\underline{\theta}, \bar{s}] : F(x) > L_{\bar{s}}(x)\}$ *is an interval.*

In every symmetric equilibrium of every rule,

$$u_F \leq \max_{\bar{s} \in [2p - \underline{\theta}, \overline{\theta}]} u_F(\bar{s}).$$

If, moreover, $I_{\bar{U}_{2p-\underline{\theta}}} < I_F < I_{\bar{U}_{\overline{\theta}}}$ on $(\underline{\theta}, \overline{\theta})$, the maximum is attained only at widths in $(2p - \underline{\theta}, \overline{\theta})$, so the best uniformizing rule strictly beats both ends of the family, the noiseless rule and the best fully revealing rule (Appendix C.8.8). Every concave prior satisfies (i)–(iii), and every non-uniform one satisfies this strict condition. Condition (iii) holds whenever F is single-peaked or U-shaped (Lemma 19).[9]

For the two priors of Example 5, the second best is 0.456 and 0.509, attained at widths $\bar{s} \approx 0.73$ and 0.94. Near full revelation, narrowing the width, hence lowering the noise, makes the higher posterior mean win more often; this raises the firm's payoff in proportion to the narrowing. Since the posterior means must now fit into a narrower uniform, the agents pool, but only their strongest types, so the loss is of smaller order. Near the noiseless rule, widening the width lets the agents spread their pooled posterior means over a wider uniform. This raises the payoff in proportion to the change, because the payoff is the slope of winning probabilities times the signal variance: the slope is inversely proportional to the width, while the variance rises with its square. At the bottom of the pool, since the posterior means can be no more dispersed than the types, this constraint binds, and a thin slice of the weakest types is exactly revealed, again at a loss of smaller order.

[9]Condition (ii) cannot be dropped, but it can be weakened (Online Appendix OA.3.9). Beta$(\frac{1}{2}, \frac{1}{2})$ satisfies (i) and (iii), and every uniformizing rule gives it at most 0.6976. Revealing the types below 0.371 and above 0.807 and pooling the types in $(0.371, 0.807)$ into posterior means spread uniformly over $[0.404, 0.788]$ gives 0.6987 in a symmetric equilibrium of a suitable rule, and no rule gives more than 0.7015 in such an equilibrium (the first best is 0.7026).

Unlike the noiseless payoff, the second best rises with dispersion (Appendix C.8.10).

Corollary 4. *Let F and F' have the same mean and satisfy (i)–(iii) of Theorem 3. If $F \leq_{cx} F'$, the second best under F' is at least the second best under F.*

For concave priors, the best uniformizing rule can be run as a *random pass mark*: the firm draws a bar at random, compares the agents exactly if the higher posterior mean clears it, and tosses a coin otherwise (Lemma OA.11). The funders' lottery of Subsection 4.4 does the reverse, tossing a coin among the agents who clear the bar, and with a continuum of types this caps the firm's payoff: whatever the distribution of the bar, $u_F = p + \mathrm{Cov}_G(s, g(s)) \leq p + \mathbb{E}_G[(s-p)^+] \leq p + \mathbb{E}_F[(\theta - p)^+]$, attained by a fixed bar at $\mathbb{E}_F[\theta \mid \theta \geq p]$; for Beta(2, 2), this gives 0.594, below both the noiseless rule and the best fully revealing rule. For the other priors of Theorem 3, such as those whose density strictly increases near $\underline{\theta}$, no pass mark works, but a lottery over coarse grades does (Appendix C.8.7). With binary types, a deterministic coarse grade attains the first best; with a continuum, it discards real distinctions between types.

5.2.4 When noise helps

Theorem 3 requires the noiseless equilibrium signal to be uniform.[10] In general, for a prior with a density that is continuous on $[\underline{\theta}, \overline{\theta}]$, the noiseless rule has a unique symmetric equilibrium (G_0, G_0); if the density has finitely many turning points, it splits the types into intervals: the types in some intervals are revealed exactly, and the types in the others are pooled into posterior means spread uniformly, in *pools* (Lemmas 20 and 21; for priors with a positive, continuously differentiable density with finitely many peaks, also Hwang et al., 2026, Theorem 1). The next result assumes this.

[10] Condition (i) cannot be dropped. For Beta(2, 5), which satisfies (ii) and (iii), every uniformizing rule gives at most 0.3677, while ranking the types below 0.030 exactly and uniformizing above gives 0.3694.

Theorem 4. *Let F have a density f that is continuous on $[\underline{\theta}, \overline{\theta}]$, and let $\underline{\theta} = z_0 < z_1 < \cdots < z_m = \overline{\theta}$ be such that, for each j, f is strictly monotone on $[z_{j-1}, z_j]$ and continuously differentiable on (z_{j-1}, z_j). The noiseless rule is optimal among general selection rules, i.e., no rule gives the firm a higher utility in any equilibrium, if and only if F is convex on its support. If F is not convex, some general selection rule nondecreasing in the own posterior mean has a symmetric equilibrium in which the firm's utility strictly exceeds that in the noiseless equilibrium.*

Why noise helps. In the noiseless equilibrium, posterior means in each pool are spread uniformly, so an agent's winning probability $V(s) = G(s)$ is linear there, and he is indifferent across the pool. Since $u_F = p + 2\,\mathrm{Cov}(s, V(s))$, the firm would gain if the pooled posterior means were more dispersed, but exact ranking rules this out: it breaks the agents' indifference. With more posterior means clustered at the ends of the pool, the winning probability would be steeper at both ends than in the middle, so no longer linear. The improving rule makes room with noise that moves a small amount ε of winning probability from the top of the pool to its bottom: the highest posterior means win slightly less and the lowest slightly more. The firm now picks the better agent slightly less often, at a cost of order ε. But now posterior means clustered at the ends win exactly at the linear rate, so V stays linear on the pool and the agents remain indifferent. They spread their pooled posterior means toward the ends, which raises $\mathrm{Cov}(s, V(s))$ by about twice that cost. At the bottom of the pool, the posterior means can be no more dispersed than the types; this constraint starts binding, and a thin slice of the weakest types is revealed exactly. This costs only order ε^2.

The assumption covers every Beta(a, b) prior with $a, b \geq 1$ other than the uniform. The proof uses only a weaker regularity condition on the noiseless equilibrium (Definition 6), which also holds for the other Beta priors and for real-analytic densities.

How much does noise gain? Figure 4 answers this for Beta(a, b) priors. Where

Theorem 3 applies ($a \leq 1 \leq b$), the noiseless equilibrium pools the weakest types, and the best uniformizing rule adds 1.6–11.4% of the first best to her payoff, closing 54–75% of the gap. Where the noiseless equilibrium reveals a bottom stretch ($a, b > 1$), the best rules found gain at most 1.5%. Noise matters most when weak types are common.

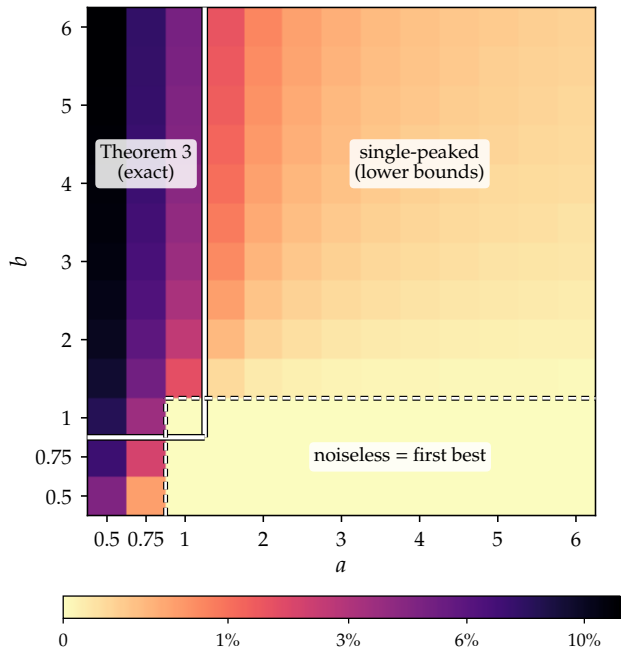


Figure 4: Gain from noise, $(u_F - u_F^0)/u_F^*$, for Beta(a, b) priors (axes not to scale): u_F is the firm's payoff under the best rule found, u_F^0 under the noiseless rule, and u_F^* the first best. Left of the solid line ($a \leq 1 \leq b$), Theorem 3 applies and u_F is the second best. Below the dashed line, F is convex and the noiseless rule attains the first best. Elsewhere, u_F is the best of the uniformizing rules and of the rules that rank the lowest types exactly and uniformize above, a lower bound on the second best.

6. Conclusion

This paper asks whether a firm choosing between two agents who design their signals should eliminate noise from her evaluation. Under well S-shaped noise, agents concede bad news and pool the rest, and the shape of the noise, summarized by a single elasticity, decides how much they reveal. The right amount of noise beats the noiseless rule and, with binary types, attains the first best at priors of at least one half. With a continuum

of types, the case for noise survives even for a designer who measures posterior means perfectly and can commit to any rule: under mild regularity conditions, her best rule blurs posterior means unless the prior is convex, and for concave priors it is a random pass mark, randomizing only among low posterior means.

Online Appendix OA.4 shows how these conclusions extend. Under the mirror-image, reverse S-shaped rules, agents reveal good news instead, and better agents can hurt the firm. Beyond the well S-shaped class, the binary-type results, including the optimality of $v^*(x)$, hold under a weaker condition, while further away agents randomize over several posteriors. With a single agent, noise works the same way, devaluing a narrow lead over an outside option, but a firm that can commit to any rule needs none: it selects the agent only on a conclusive realization. With $n > 2$ agents, a uniform bonus to one randomly drawn agent makes full revelation strictly dominant, while uniform noise of unit width attains the first best only if $p \geq \frac{1}{n}$; with a general prior, the best fully revealing rule is noiseless below a cutoff (Online Appendix OA.5).

References

Au, P. H. and K. Kawai (2020). Competitive information disclosure by multiple senders. *Games and Economic Behavior 119*, 56–78.

Ball, I. (2025). Scoring strategic agents. *American Economic Journal: Microeconomics 17*(1), 97–129.

Banerjee, S., K. Munagala, Y. Shen, and K. Wang (2026). The price of competitive information disclosure. Working paper, arXiv:2504.10459.

Bergemann, D., T. Gan, and Y. Li (2023). Managing persuasion robustly: The optimality of quota rules. Working paper, arXiv:2310.10024.

Blackwell, D. (1953). Equivalent comparisons of experiments. *The Annals of Mathematical Statistics 24*(2), 265–272.

Blume, A., O. J. Board, and K. Kawamura (2007). Noisy talk. *Theoretical Economics 2*(4), 395–440.

Board, S. and J. Lu (2018). Competitive information disclosure in search markets. *Journal of Political Economy 126*(5), 1965–2010.

Boleslavsky, R. and C. Cotton (2015). Grading standards and education quality. *American Economic Journal: Microeconomics 7*(2), 248–279.

Boleslavsky, R. and C. Cotton (2018). Limited capacity in project selection: Competition through evidence production. *Economic Theory 65*(2), 385–421.

Boleslavsky, R. and C. Li (2026). Evaluating university graduates: Prestige versus merit. *Journal of Economics & Management Strategy*. Forthcoming.

Chan, W., H. Li, and W. Suen (2007). A signaling theory of grade inflation. *International Economic Review 48*(3), 1065–1090.

Che, Y.-K. and I. Gale (2000). Difference-form contests and the robustness of all-pay auctions. *Games and Economic Behavior 30*(1), 22–43.

Curello, G. and L. Sinander (2025). The comparative statics of persuasion. Working paper, arXiv:2204.07474.

Ding, B., Y. Feng, C.-J. Ho, W. Tang, and H. Xu (2023). Competitive information design for Pandora's box. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM.

Drugov, M. and D. Ryvkin (2020). How noise affects effort in tournaments. *Journal of Economic Theory 188*, 105065.

Dworczak, P. and G. Martini (2019). The simple economics of optimal persuasion. *Journal of Political Economy 127*(5), 1993–2048.

Ederer, F., R. Holden, and M. Meyer (2018). Gaming and strategic opacity in incentive provision. *RAND Journal of Economics 49*(4), 819–854.

Fang, D. and T. Noe (2022). Less competition, more meritocracy? *Journal of Labor Economics 40*(3), 669–701.

Fang, F. C. and A. Casadevall (2016). Research funding: the case for a modified lottery. *mBio 7*(2), e00422–16.

Frankel, A. and N. Kartik (2022). Improving information from manipulable data. *Journal of the European Economic Association 20*(1), 79–115.

Gao, E. and D. Luo (2026). (Robust) information acquisition design. Working paper, arXiv:2312.02465.

Gentzkow, M. and E. Kamenica (2016). A Rothschild-Stiglitz approach to Bayesian persuasion. *American Economic Review 106*(5), 597–601.

Gentzkow, M. and E. Kamenica (2017). Competition in persuasion. *The Review of Economic Studies 84*(1), 300–322.

Grossman, S. J. and J. E. Stiglitz (1980). On the impossibility of informationally efficient markets. *American Economic Review 70*(3), 393–408.

Harbaugh, R. and E. Rasmusen (2018). Coarse grades: Informing the public by withholding information. *American Economic Journal: Microeconomics 10*(1), 210–235.

Heyard, R., M. Ott, G. Salanti, and M. Egger (2022). Rethinking the funding line at the Swiss National Science Foundation: Bayesian ranking and lottery. *Statistics and Public Policy 9*(1), 110–121.

Hirshleifer, J. (1989). Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. *Public Choice 63*(2), 101–112.

Hoppe, H. C., B. Moldovanu, and A. Sela (2009). The theory of assortative matching based on costly signals. *Review of Economic Studies 76*(1), 253–281.

Hvide, H. K. and E. G. Kristiansen (2003). Risk taking in selection contests. *Games and Economic Behavior 42*(1), 172–179.

Hwang, I., K. Kim, and R. Boleslavsky (2026). Competitive advertising and pricing. *The Review of Economic Studies*. Forthcoming.

Jain, V. and M. Whitmeyer (2026). Competitive disclosure of information to a rationally inattentive agent. *Journal of Economic Behavior & Organization 242*, 107410.

Kamenica, E. (2019). Bayesian persuasion and information design. *Annual Review of Economics 11*, 249–272.

Kamenica, E. and M. Gentzkow (2011). Bayesian persuasion. *American Economic Review 101*(6), 2590–2615.

Kleiner, A., B. Moldovanu, and P. Strack (2021). Extreme points and majorization: Economic applications. *Econometrica 89*(4), 1557–1593.

Kolotilin, A. (2018). Optimal information disclosure: A linear programming approach. *Theoretical Economics 13*(2), 607–635.

Kolotilin, A., T. Mylovanov, and A. Zapechelnyuk (2022). Censorship as optimal persuasion. *Theoretical Economics 17*(2), 561–585.

Kolotilin, A., T. Mylovanov, A. Zapechelnyuk, and M. Li (2017). Persuasion of a privately informed receiver. *Econometrica 85*(6), 1949–1964.

Lazear, E. P. and S. Rosen (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy 89*(5), 841–864.

Letina, I., S. Liu, and N. Netzer (2023). Optimal contest design: Tuning the heat. *Journal of Economic Theory 213*, 105616.

Morgan, J., J. Tumlinson, and F. Várdy (2022). The limits of meritocracy. *Journal of Economic Theory 201*, 105414.

Muratov, O. (2025). Mapping an information design game into an all-pay auction. *Review of Economic Design 29*(3), 513–543.

Niculescu, C. P. and L.-E. Persson (2003). Old and new on the Hermite–Hadamard inequality. *Real Analysis Exchange 29*(2), 663–686.

Ostrovsky, M. and M. Schwarz (2010). Information disclosure and unraveling in matching markets. *American Economic Journal: Microeconomics 2*(2), 34–63.

Perez-Richet, E. and V. Skreta (2022). Test design under falsification. *Econometrica 90*(3), 1109–1142.

Pier, E. L., M. Brauer, A. Filut, A. Kaatz, J. Raclaw, M. J. Nathan, C. E. Ford, and M. Carnes (2018). Low agreement among reviewers evaluating the same NIH grant applications. *Proceedings of the National Academy of Sciences 115*(12), 2952–2957.

Popov, S. V. and D. Bernhardt (2013). University competition, grading standards, and grade inflation. *Economic Inquiry 51*(3), 1764–1778.

Prat, A. (2005). The wrong kind of transparency. *American Economic Review 95*(3), 862–877.

Prendergast, C. and R. H. Topel (1996). Favoritism in organizations. *Journal of Political Economy 104*(5), 958–978.

Röbbecke, M. and D. Simon (2023). Riskante Forschung und teilrandomisierte Begutachtungsverfahren: Neue Wege der Förderlinie "Experiment!" der VolkswagenStiftung. *Beiträge zur Hochschulforschung 45*(2), 8–30.

Scharfstein, D. S. and J. C. Stein (2000). The dark side of internal capital markets: Divisional rent-seeking and inefficient investment. *The Journal of Finance 55*(6), 2537–2564.

Tsakas, E. and N. Tsakas (2021). Noisy persuasion. *Games and Economic Behavior 130*, 44–61.

Tsakas, E., N. Tsakas, and D. Xefteris (2021). Resisting persuasion. *Economic Theory 72*(3), 723–742.

Tullock, G. (1980). Efficient rent seeking. In J. M. Buchanan, R. D. Tollison, and G. Tullock (Eds.), *Toward a Theory of the Rent-Seeking Society*, pp. 97–112. College Station: Texas A&M University Press.

Overview of the Appendices

Appendix A proves the results of Section 3. Appendix B gives the equilibria of the noiseless rule, proves the results of Section 4, and constructs the equilibrium for the triangular selection rules of Subsection 4.2 in all but one case. Appendix C proves the results of Subsection 4.4 and Section 5, starting with best replies and censorship under an arbitrary prior. The Online Appendix contains the remaining results, numbered with the prefix OA. Section OA.1 gives the limit of pooling, the regions and payoffs under unequal priors, and how informativeness and selection reversals respond to the prior; Section OA.2 treats costly signals, the remaining triangular case and the design of rules for unequal priors; Section OA.3 refines the results of Section 5, including

the best fully revealing rules, the random pass mark, continuous rules, the noiseless equilibrium and a weaker condition (ii) of Theorem 3; Section OA.4 contains the extensions to reverse S-shaped rules, the boundary of the well S-shaped class, the case of a single agent and rules generated by i.i.d. shocks; and Section OA.5 lets more than two agents compete, with binary types and with a continuum of types.

A. Proofs for Section 3

A.1 Properties of well S-shaped rules

Throughout the proofs, I use the following properties of a well S-shaped $v(x)$.

1) *Regularity.* Since $v'(x)$ is concave, it is continuous on $(-1,1)$ and has one-sided derivatives there; I set $v'(\pm 1)$ equal to its one-sided limits. I write $v''(x)$ and $\tilde{v}''(x)$ for either one-sided derivative, and every statement involving them holds for both.

2) *Shape.* Since $v(x)$ is increasing, $v'(1) \geq 0$, and since $v'(x)$ is even, concave and uniquely maximized at 0, $v'(x) \geq (1-|x|)v'(0) + |x|v'(1) > v'(1) \geq 0$ on $(-1,1)$, so $v(x)$ is strictly increasing. For $0 \leq x < y \leq 1$, the same argument gives $v'(x) \geq \frac{y-x}{y}v'(0) + \frac{x}{y}v'(y) > v'(y)$, so $v'(x)$ is strictly decreasing on $[0,1]$. Since $\tilde{v}'(x) = \frac{v'(x)-v'(0)}{|v(1)-\frac{1}{2}-v'(0)|}$, it is even, concave, and strictly decreasing on $[0,1]$ with $\tilde{v}'(0) = 0$. So $\tilde{v}(x)$ is negative, strictly decreasing and strictly concave on $(0,1]$. By the concavity of $\tilde{v}'(x)$, $\tilde{v}''(x) \leq \frac{\tilde{v}'(x)}{x} < 0$ on $(0,1)$, and $-\tilde{v}''(x)$ is non-decreasing on $(0,1)$.

3) *Elasticity.* The following lemma bounds the elasticity of $\tilde{v}(x)$ from below.

Lemma 3. *Let $v(x)$ be well S-shaped. Then $\mathcal{E}_{\tilde{v}}(x) \geq 2$ for all $x \in (0,1]$, with equality at x only if $\tilde{v}'(t)$ is linear on $[0,x]$. Hence $\frac{\tilde{v}(x)}{x^2}$ is non-increasing on $(0,1]$, so $\tilde{v}(x) \geq -x^2$ on $[0,1]$; if $v'(x)$ is not linear near 0, the ratio is strictly decreasing.*

Proof. Let $x \in (0,1]$ and $\phi := -\tilde{v}'$, which is convex on $[0,x]$ with $\phi(0) = 0$ by 2), so $\phi(t) \leq \frac{t}{x}\phi(x)$ for $t \in [0,x]$. Hence $-\tilde{v}(x) = \int_0^x \phi(t)dt \leq \frac{x\phi(x)}{2}$, i.e., $\mathcal{E}_{\tilde{v}}(x) \geq 2$, with

equality only if ϕ is linear on $[0, x]$. Finally, $\left(\frac{\tilde{v}(x)}{x^2}\right)' = \frac{\tilde{v}(x)(\mathcal{E}_{\tilde{v}}(x)-2)}{x^3} \leq 0$, with strict inequality if $v'(x)$ is not linear near 0, and $\tilde{v}(1) = -1$. □

4) *Non-linearity near* 0. If $v'(x)$ is not linear near 0, then it is not affine on any interval whose closure contains 0: on an interval with 0 in its interior, an affine $v'(x)$ would be constant, contradicting the unique maximum at 0, and an affine piece ending at 0 gives, by evenness, one starting at 0. Hence, for every $G \in \mathcal{G}(p)$, $\overline{v}'_G(s) := \int_0^1 v'(s-t)dG(t)$ is not affine on any interval $[\alpha, \beta]$ with $G([\alpha, \beta]) > 0$. Indeed, $\overline{v}'_G$ is a mixture of the concave functions $v'(s-t)$, so it is affine on $[\alpha, \beta]$ only if $v'(x)$ is affine on $[\alpha - t, \beta - t]$ for G-almost every t, and the closure of this interval contains 0 whenever $t \in [\alpha, \beta]$.
5) *Symmetric bad news equilibria.* The next lemma verifies best replies and uniqueness in the symmetric equilibrium of Theorem 1.

Lemma 4. *Let $v(x)$ be well S-shaped, the priors (p, p), and $a \in (p, 1]$.*

(i) If $a < 1$ and $p = a - \frac{\tilde{v}(a)}{\tilde{v}'(a)}$, or $a = 1$ and $p \geq \overline{p}$, then $(\{0, a\}; \{0, a\})$ is the unique equilibrium.

(ii) If $a < 1$ and $p = a - \frac{\tilde{v}(a)}{\tilde{v}'(a)}$, then $p < a \leq 2p$, and $a < 2p$ if $v'(x)$ is not linear near 0.

Proof. Uniqueness. As the game is symmetric and constant-sum, its equilibria form a product set $E \times E$ (Section 2). If $G^* \in E$ is its own unique best reply, then $E = \{G^*\}$, as every $G \in E$ is a best reply to G^*.

Best reply to $\{0, a\}$. Let agent j play $\{0, a\}$ with $a \in (p, 1]$, and let $\gamma := \frac{a-p}{a} \in (0, 1)$. Then agent i's normalized value function is

$$\overline{v}(s) = \gamma \tilde{v}(s) + (1 - \gamma)\tilde{v}(s - a).$$

Let $l(s)$ be the line through $(0, \overline{v}(0))$ and $(a, \overline{v}(a))$, whose slope is $\frac{\tilde{v}(a)}{a}$, and $d(s) :=$

$l(s) - \overline{v}(s)$ on $[0, a]$, so that $d(0) = d(a) = 0$. As $\tilde{v}'(x)$ is even,

$$d'(s) = \frac{\tilde{v}(a)}{a} - \gamma\tilde{v}'(s) - (1-\gamma)\tilde{v}'(a-s),$$

which is convex on $[0, a]$, since $\tilde{v}'(x)$ is concave. If $d'(a) \leq 0$, then $d \geq 0$ on $[0, a]$: with $z := \inf\{s \in [0, a] : d'(s) \leq 0\}$, convexity gives $d' \leq 0$ on $[z, a]$, while $d' > 0$ on $[0, z)$. If $a < 1$ and $d'(a) = 0$, then for $s \in (a, 1]$,

$$\overline{v}'(s) = \gamma\tilde{v}'(s) + (1-\gamma)\tilde{v}'(s-a) < \gamma\tilde{v}'(a) + (1-\gamma)\tilde{v}'(0) = \overline{v}'(a),$$

as $\tilde{v}'(x)$ is strictly decreasing on $[0, 1]$, so $\overline{v}$ lies strictly below its tangent l on $(a, 1]$. Hence, if $d'(a) \leq 0$ for $a = 1$, or $d'(a) = 0$ for $a < 1$, then $\{0, a\}$ is a best reply to itself. Since $\overline{v}'(a) = \gamma\tilde{v}'(a)$,

$$d'(a) = 0 \;\Leftrightarrow\; \gamma\tilde{v}'(a) = \frac{\tilde{v}(a)}{a} \;\Leftrightarrow\; p = a - \frac{\tilde{v}(a)}{\tilde{v}'(a)},$$

where the last step uses $\gamma a = a - p$.

Step 1: other best replies. Suppose that $d'(a) \leq 0$ and $d(c) = 0$ for some $c \in (0, a)$. Then c minimizes $d \geq 0$ on $[0, a]$, so $d'(c) = 0$; the convex d' is then ≤ 0 on $[c, a]$, and $\int_c^a d' = 0$ gives $d' \equiv 0$ there; by convexity again $d' \geq 0$ on $[0, c]$, and $d(0) = d(c)$ gives $d' \equiv 0$ on $[0, a]$, so in particular $d'(a) = 0$. So $\gamma\tilde{v}'(s) + (1-\gamma)\tilde{v}'(a-s)$ is affine on $[0, a]$; as a sum of two concave functions, it is affine only if both are, so $\tilde{v}'(s) = -Ks$ on $[0, a]$ for some $K > 0$, and $d' \equiv 0$ then requires $K(1-2\gamma) = 0$, i.e., $\frac{p}{a} = 1 - \gamma = \frac{1}{2}$. Hence, either $d > 0$ on $(0, a)$, and $\{0, a\}$ is the unique best reply to itself, or $\tilde{v}(x) = -\frac{K}{2}x|x|$ on $[-a, a]$ and $p = \frac{a}{2}$, and every $G \in \mathcal{G}(p)$ supported on $[0, a]$ is a best reply to $\{0, a\}$.

Step 2: the quadratic case. Take such a $G \neq \{0, a\}$ and let $w(s) := \int_0^a \tilde{v}(s-t)dG(t)$ be the normalized value function against it. For $s \in [0, a]$, $w'(s) = -K\int_0^a |s-t|dG(t)$,

hence $w'(a) = -K(a-p)$, while $w(0) = \frac{K}{2}\mathbb{E}_G[t^2]$ and $w(a) = -\frac{K}{2}\mathbb{E}_G[(a-t)^2]$. So the secant slope from $(0, w(0))$ to $(a, w(a))$ exceeds $w'(a)$ by

$$\frac{K}{a}\Big(\frac{a^2}{2} - \mathbb{E}_G[t^2]\Big) > \frac{K}{a}\Big(\frac{a^2}{2} - ap\Big) = 0,$$

since $t^2 \leq at$ on $[0, a]$, with equality only at $t \in \{0, a\}$, and $p = \frac{a}{2}$. Hence $w(s)$ lies strictly above the secant for $s < a$ close enough to a, so, for such $s > p$, $\{0, s\}$ yields a strictly higher payoff against G than $\{0, a\}$. Thus $\{0, a\}$ is not a best reply to G, $G \notin E$, and so the equilibrium is unique.

(i) If $a < 1$, the condition gives $d'(a) = 0$, so $\{0, a\}$ is a best reply to itself, and Steps 1 and 2 give uniqueness. For $a = 1$, $\gamma = 1 - p$ and $d'(1) = -1 + (1-p)\mathcal{E}_{\tilde{v}}(1)$, so $d'(1) \leq 0$ if and only if $p \geq \overline{p}$, and then $\{0, 1\}$ is a best reply to itself. If $p > \overline{p}$, then $d'(1) < 0$, so $d > 0$ on $(0, 1)$ by Step 1, $\{0, 1\}$ is the unique best reply to itself, and the equilibrium is unique. If $p = \overline{p}$, then $d'(1) = 0$, and Steps 1 and 2 with $a = 1$ give uniqueness.

(ii) Since $\frac{p}{a} = 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)}$, and $\mathcal{E}_{\tilde{v}}(a) \geq 2$ by Lemma 3, with strict inequality if $v'(x)$ is not linear near 0, $p < a \leq 2p$, and $a < 2p$ in the latter case. □

A.2 Proof of Lemma 1

(i) As shown in the text, $\overline{v}'(s_i) = \int_0^1 v'(s_i - s_j)dG_j(s_j)$ is concave on $[0, 1]$. For a maximizer c of $\overline{v}'$, $\overline{v}$ is convex on $[0, c]$ and concave on $[c, 1]$.

(ii) Let $slp(s) := \frac{\overline{v}(s) - \overline{v}(0)}{s}$ for $s \in (0, 1]$ and $slp(0) := \overline{v}'(0)$, so that a is the largest maximizer of slp on $[0, 1]$, which exists as slp is continuous. For $s \in (0, 1]$, differentiating gives $slp'(s) = \frac{N(s)}{s^2}$, where

$$N(s) := s\overline{v}'(s) - \overline{v}(s) + \overline{v}(0) = \int_0^s \big(\overline{v}'(s) - \overline{v}'(t)\big)\, dt.$$

By (i), $N(s) \geq 0$ on $[0, c]$, and $N(s)$ is non-increasing on $[c, 1]$, since for $c \leq s_1 < s_2 \leq 1$,

$$N(s_2) - N(s_1) = s_1 \left(\overline{v}'(s_2) - \overline{v}'(s_1)\right) + \int_{s_1}^{s_2} \left(\overline{v}'(s_2) - \overline{v}'(t)\right) dt \leq 0.$$

Hence slp is non-decreasing on $[0, c]$ and changes its monotonicity on $[c, 1]$ at most once, from non-decreasing to non-increasing. So $a \geq c$, and $\overline{v}$ is concave on $[a, 1]$. Let $l(s) := \overline{v}(0) + s \cdot slp(a)$. Then $l \geq \overline{v}$ on $[0, a]$ with equality at 0 and a, and if $a < 1$, $N(a) = 0$, i.e., l is tangent to $\overline{v}$ at a. Therefore $\overline{v}^{cv}$ equals l on $[0, a]$ and $\overline{v}$ on $[a, 1]$. By Kamenica and Gentzkow (2011, Corollary 2), a signal is optimal if and only if it is supported where $\overline{v}$ touches a line supporting $\overline{v}^{cv}$ at p_i. So $\{0, a\}$ is optimal if $p_i < a$, and $\{p_i\}$ is optimal if $p_i \geq a$, where $\overline{v}^{cv}(p_i) = \overline{v}(p_i)$.

If $v'(x)$ is not linear near 0, the concave $\overline{v}'$ is strictly increasing on $[0, z_1]$, constant on $[z_1, z_2]$ and strictly decreasing on $[z_2, 1]$ for some $z_1 \leq z_2$, and $z_1 > 0$ unless $z_2 = 0$: if $\overline{v}'$ were constant on $[0, z_2]$ with $z_2 > 0$, then $G_j([0, z_2]) = 0$ by property 4) in Appendix A.1, so $\overline{v}''(s) = \int_{z_2}^{1} v''(s - s_j) dG_j(s_j) > 0$ on $(0, z_2)$, as $v'' > 0$ on $[-1, 0)$ by property 2), a contradiction. Hence $N > 0$ on $(0, z_2]$ and N is strictly decreasing on $[z_2, 1]$, so $\overline{v} < l$ on $(0, a)$, and $\overline{v}$ is strictly concave on $[a, 1]$. Hence the supporting line at p_i touches $\overline{v}$ only on $\{0, a\}$ if $p_i \leq a$ and only at p_i if $p_i > a$, so the optimum is unique.

A.3 Proof of Proposition 1

Let $\overline{v}(s_i)$ denote the value function of agent i for $\tilde{v}(x)$ rather than $v(x)$, which does not change best replies. If agent j plays $\{0, 1\}$, then $\overline{v}(s_i) = (1 - p_j)\tilde{v}(s_i) - p_j\tilde{v}(1 - s_i)$, which is S-shaped by Lemma 1, with $\overline{v}(1) - \overline{v}(0) = \tilde{v}(1) = -1$ and $\overline{v}'(1) = (1 - p_j)\tilde{v}'(1)$. By Lemma 1, $\{0, 1\}$ is a best reply if and only if $a = 1$, i.e., if and only if $\overline{v}$ lies below its secant $L(s) := \overline{v}(0) - s$ on $[0, 1]$. This implies $\overline{v}'(1) \geq -1$. Conversely, if $a < 1$, then $\overline{v}$ is concave on $[a, 1]$ by the proof of Lemma 1, so $\overline{v}(a) \leq \overline{v}(1) - (1 - a)\overline{v}'(1)$, while

$slp(a) > slp(1)$ gives $\overline{v}(a) > \overline{v}(0) - a$; together, $\overline{v}'(1) < -1$. Hence

$$\{0,1\} \text{ is a best reply } \Leftrightarrow\ (1-p_j)\tilde{v}'(1) \geq -1 \ \Leftrightarrow\ p_j \geq 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)} = \overline{p},$$

and $(\{0,1\};\{0,1\})$ is an equilibrium if and only if $p_1, p_2 \geq \overline{p}$.

For uniqueness, since all equilibria form a product set, it suffices that $\{0,1\}$ is the unique best reply to $\{0,1\}$ for both agents. Another best reply exists only if $D := \overline{v} - L \leq 0$ vanishes at some $s_0 \in (0,1)$. Since D is convex on $[0,c]$ and concave on $[c,1]$ for some c, it then vanishes on $[s_0,1]$: a convex D with an interior maximum at $s_0 < c$ vanishes on $[0,c]$, and a concave D that vanishes at both ends of $[\max(s_0,c),1]$ is non-negative there. Hence $\overline{v}'(1) = -1$, i.e., $p_j = \overline{p}$, and $\overline{v}$ is affine on $[s_0,1]$. With $w := -\tilde{v}''$, which is non-decreasing on $(0,1)$, $\overline{v}''(s) = p_j w(1-s) - (1-p_j)w(s) = 0$ on $(s_0,1)$, and letting $s \to 1$ gives $\frac{p_j}{1-p_j} = \frac{w(1-)}{w(0+)}$, where $w(0+) > 0$, as otherwise $w(1-) = 0$ and $w \equiv 0$. But $\overline{p} = 1 - \frac{1}{\mathcal{E}_{\tilde{v}}(1)} = \frac{\int_0^1 t w(t)dt}{\int_0^1 w(t)dt}$, and pairing t with $1-t$,

$$w(1-)\int_0^1 (1-t)w - w(0+)\int_0^1 tw = \int_0^{\frac{1}{2}} \Big[(1-t)\big(w(1-)w(t) - w(0+)w(1-t)\big) + t\big(w(1-)w(1-t) - w(0+)w(t)\big)\Big]$$

as $w(0+) \leq w(t) \leq w(1-t) \leq w(1-)$ for $t \leq \frac{1}{2}$, with equality only if w is constant. So $\frac{\overline{p}}{1-\overline{p}} < \frac{w(1-)}{w(0+)}$ unless $\tilde{v}(x) = -x^2$ on $[0,1]$, and then $\overline{p} = \frac{1}{2}$. This proves uniqueness unless $\tilde{v}(x) = -x^2$ on $[0,1]$ and $\min(p_1,p_2) = \frac{1}{2}$. If $p_1 = p_2 = \frac{1}{2}$, the uniqueness follows from Lemma 4 with $a = 1$.

A.4 Proof of Theorem 1

If $p \geq \overline{p}$, full revelation is an equilibrium by Proposition 1, and it is unique for $p > \overline{p}$ by Proposition 1 and for $p = \overline{p}$ by Lemma 4. So let $p < \overline{p}$, and let $P(a) := a - \frac{\tilde{v}(a)}{\tilde{v}'(a)} = a\big(1 - \frac{1}{\mathcal{E}_{\tilde{v}}(a)}\big)$ on $(0,1]$, which is continuous, as $\tilde{v}' < 0$ on $(0,1]$. On $(0,1)$, P has both

one-sided derivatives, $\frac{\tilde{v}(a)\tilde{v}''(a)}{\tilde{v}'(a)^2} > 0$, as $\tilde{v} < 0$ and $\tilde{v}'' < 0$ there by property 2), so P is strictly increasing, as is every continuous function with a positive right derivative on an interval. Moreover, $P(1) = \overline{p}$, and $P(a) \in [\frac{a}{2}, a)$ by Lemma 3, so $P(a) \to 0$ as $a \to 0_+$. Hence P maps $(0, 1]$ continuously and strictly increasingly onto $(0, \overline{p}]$, and the equation of the statement has a unique solution $a(p) \in (p, 1)$, which is continuous and strictly increasing in p. By Lemma 4(i), $(\{0, a\}; \{0, a\})$ is the unique equilibrium, and $\mathbb{P}(s_i = 0) = \frac{a-p}{a} = \frac{1}{\mathcal{E}_{\tilde{v}}(a)}$.

A.5 Pooling by both agents

Lemma 5. *Let $v(x)$ be well S-shaped. Then $(\{p_1\}; \{p_2\})$ is never an equilibrium.*

Proof. Against $\{p_j\}$, $\overline{v}(s) = v(s - p_j)$, so $\overline{v}'(s) = v'(s - p_j)$ is strictly increasing on $[0, p_j]$, since $v'(x)$ is even and strictly decreasing on $[0, 1]$ by property 2) in Appendix A.1. As in the proof of Lemma 1, let $slp(s) := \frac{\overline{v}(s)-\overline{v}(0)}{s}$, so that $slp'(s) = \frac{N(s)}{s^2}$ with $N(s) = \int_0^s (\overline{v}'(s) - \overline{v}'(t))dt$. Then $N > 0$ on $(0, p_j]$, and, by continuity, slp is strictly increasing on $(0, s]$ for some $s \in (p_j, 1]$. If $p_i \leq p_j$, the signal $\{0, s\}$ with mean p_i yields $\overline{v}(0) + p_i\, slp(s) > \overline{v}(0) + p_i\, slp(p_i) = \overline{v}(p_i)$, so $\{p_i\}$ is not a best reply. Hence $\{p_i\}$ is a best reply only if $p_i > p_j$, which cannot hold for both agents.

A.6 Proof of Proposition 2

If $p_2 \geq \overline{p}$, the claim follows from Proposition 1, so let $p_2 < \overline{p}$.

Preliminaries. For a signal of agent j let $N(s) := s\overline{v}'(s) - \overline{v}(s) + \overline{v}(0)$, so that $slp'(s) = N(s)/s^2$. By the proof of Lemma 1 (ii), $N > 0$ on $(0, a)$ and $N < 0$ on $(a, 1]$, with a from Lemma 1. Hence agent i plays $\{p_i\}$ if and only if $N(p_i) \leq 0$, and $\{0, 1\}$ if

and only if $N(1) \geq 0$. For $y, c \in [0,1]$, let

$$A(y) := \tilde{v}(y) - y\tilde{v}'(y) = -\int_0^y x\tilde{v}''(x)dx,$$
$$B(y;c) := y\tilde{v}'(c-y) + \tilde{v}(c-y) - \tilde{v}(c) = -\int_0^y x\tilde{v}''(c-x)dx.$$

Since $-\tilde{v}'' > 0$ on $(0,1]$, $A(y) > 0$, and $B(y;c) > 0$ for $0 < y \leq c$. If agent j plays $\{0,c\}$, so that $\mathbb{P}(s_j = c) = q := \frac{p_j}{c}$, then $\overline{v}(s) = (1-q)\tilde{v}(s) + q\tilde{v}(s-c)$ and

$$N(s) = -(1-q)A(s) + qB(s;c); \tag{N}$$

pooling is the case $c = p_j$, $q = 1$.

Step 1: $b(\cdot)$ and p^*. Against $\{p_1\}$, $N(s) = B(s;p_1)$. Hence agent 2 never plays $\{p_2\}$, as $p_2 \leq p_1$, and plays $\{0, b(p_1)\}$, where $b(p_1)$ solves $B(b;p_1) = 0$, i.e. $b\tilde{v}'(b-p_1) = \tilde{v}(b-p_1) + \tilde{v}(p_1)$, or $b(p_1) = 1$ if $B(1;p_1) \geq 0$. With $b = p_1 + \tilde{b}$ this equation reads

$$\varphi(\tilde{b}) := \tilde{v}(\tilde{b}) - (p_1 + \tilde{b})\tilde{v}'(\tilde{b}) + \tilde{v}(p_1) = 0,$$

where $\varphi'(\tilde{b}) = -(p_1 + \tilde{b})\tilde{v}''(\tilde{b}) > 0$, $\varphi(0) = \tilde{v}(p_1) < 0 < 2A(p_1) = \varphi(p_1)$ and $\partial_{p_1}\varphi = \tilde{v}'(p_1) - \tilde{v}'(\tilde{b}) < 0$ for $\tilde{b} < p_1$. So the root is unique, $p_1 < b(p_1) < 2p_1$, and b is continuous, strictly increasing while $b < 1$. Let p^* be the smallest p_1 with $b(p_1) = 1$, solving $\tilde{v}(1-p^*) + \tilde{v}(p^*) = \tilde{v}'(1-p^*)$; it exists and $p^* \in (\frac{1}{2}, 1)$, since $b(\frac{1}{2}) < 1$ and $B(1;1) = 1 > 0$. Thus against $\{x\}$ the other agent plays $\{0,1\}$ if and only if $B(1;x) \geq 0$, i.e. $x \geq p^*$, as b is non-decreasing.

Step 2: $a(\cdot)$. Against $\{0,1\}$, by (N),

$$N(s) = \big(A(s) + B(s;1)\big)\big(p_2 - \Phi(s)\big), \qquad \Phi := \frac{A}{A + B(\cdot\,;1)}.$$

Φ is strictly increasing: writing $w = -\tilde{v}''$,

$$\partial_y B(y;1)\,A(y) - B(y;1)\,A'(y) = y\int_0^y x\big[w(1-y)w(x) - w(1-x)w(y)\big]dx < 0,$$

since $w(x) \le w(y)$ and $w(1-y) \le w(1-x)$ for $x < y$, and $w(x) < w(y)$ for x close to 0, as $v'(x)$ is not linear near 0. Moreover, $\Phi(1) = \frac{|\tilde{v}'(1)|-1}{|\tilde{v}'(1)|} = \overline{p}$. Hence agent 1 plays $\{p_1\}$ if $p_1 \ge a(p_2)$ and $\{0, a(p_2)\}$ otherwise, where $a(p_2) = 0$ if $p_2 \le \Phi(0^+)$, $a(p_2) = \Phi^{-1}(p_2)$ if $\Phi(0^+) < p_2 < \overline{p}$, and $a(p_2) = 1$ if $p_2 \ge \overline{p}$; a is continuous and non-decreasing, since Φ is continuous and strictly increasing.

Step 3: $\check{p}_2(\cdot)$ and $\hat{p}_1(\cdot)$. By (N), against $\{0, b(p_1)\}$, agent 1 plays $\{p_1\}$ if and only if

$$p_2 \le \check{p}_2(p_1) := \frac{b(p_1)A(p_1)}{A(p_1) + B\big(p_1; b(p_1)\big)},$$

and against $\{0, a\}$ with $a > p_1$, agent 2 plays $\{0,1\}$ if and only if $-(a-p_1)A(1) + p_1 B(1;a) \ge 0$. By Step 1, $B(1;a) \ge 0$ if and only if $a \ge p^*$, so for $a < p^*$ this fails, and for $a \ge p^*$ it reads $p_1 \ge \frac{aA(1)}{A(1)+B(1;a)}$. For $a = a(p_2) \ge p^*$ denote this bound by $\hat{p}_1(p_2)$.

Step 4: cases (i)–(iii). By Steps 1 and 2, $(\{p_1\};\{0,1\})$ is an equilibrium if and only if $p_1 \ge p^*$ and $p_1 \ge a(p_2)$; by Steps 1 and 3, $(\{p_1\};\{0,b\})$ with $b < 1$ is an equilibrium if and only if $b = b(p_1)$, $p_1 < p^*$ and $p_2 \le \check{p}_2(p_1)$; by Steps 2 and 3, $(\{0,a\};\{0,1\})$ is an equilibrium if and only if $a = a(p_2) \ge p^*$ and $\hat{p}_1(p_2) \le p_1 < a(p_2)$. An equilibrium exists by the minimax theorem, and as the game is constant-sum, all equilibria form a product set. By Lemma 1 every best reply is unique, hence so is the equilibrium.

Step 5: case (iv). For equal priors the game is symmetric, so the unique equilibrium is symmetric and none of the forms in (i)–(iii) is an equilibrium. By Step 4 this gives: $a(p) > p$ if $p \ge p^*$; $\check{p}_2(p) < p$ if $p < p^*$; and $\hat{p}_1(p) > p$ if $a(p) \ge p^*$ and $p < \overline{p}$, since then $p < a(p)$. Now let $p_2 \le p_1$ and suppose that none of (i)–(iii) holds. By Lemma 1 each agent plays $\{p_i\}$, $\{0,1\}$ or $\{0,c\}$ with $p_i < c < 1$, and by Lemma 5 not both pool at

their priors. Apart from (i)–(iv), this leaves the following forms, and each requires, by Steps 1–3 with the roles of the agents exchanged (the functions b, a, $\check{p}_2$ and $\hat{p}_1$ stay the same, only their arguments change), a condition incompatible with $p_2 \leq p_1$:

- $(\{0, b(p_2)\}; \{p_2\})$ requires $p_2 < p^*$ and $p_1 \leq \check{p}_2(p_2) < p_2$;
- $(\{0,1\}; \{p_2\})$ requires $p_2 \geq p^*$ and $p_2 \geq a(p_1) \geq a(p_2) > p_2$;
- $(\{0,1\}; \{0, a(p_1)\})$ with $a(p_1) < 1$ requires $a(p_1) \geq p^*$, $p_1 < \overline{p}$ and $p_2 \geq \hat{p}_1(p_1) > p_1$;
- $(\{0,1\}; \{0,1\})$ requires $p_2 \geq \overline{p}$ by Proposition 1.

Hence the equilibrium is $(\{0, a\}; \{0, b\})$ with $p_1 < a < 1$ and $p_2 < b < 1$.

Step 6: $a \leq b$ and continuity. The correspondence $p \mapsto \mathcal{G}(p)$ is continuous: upper hemicontinuous since the mean is weakly continuous, and lower hemicontinuous since any $G \in \mathcal{G}(p)$ is approximated by mixtures of G with δ_0 or δ_1. As payoffs are continuous, the equilibrium correspondence has a closed graph, and being single-valued with values in a compact set, it is continuous; so the equilibrium of Proposition 2 is continuous in (p_1, p_2). If $a = b = c$ in (iv), then, since $B(c; c) = -\tilde{v}(c)$, the indifference conditions $N(c) = 0$ of the two agents, with $q = \frac{p_2}{c}$ and $q = \frac{p_1}{c}$, read $\frac{c}{p_2} - 1 = -\frac{\tilde{v}(c)}{A(c)} = \frac{c}{p_1} - 1$, so $p_1 = p_2$; conversely $a = b$ on the diagonal by symmetry. On the convex set $U = \{0 < p_2 < p_1 < 1,\ p_2 < \overline{p}\}$, let R be the set where (iv) holds. R is open in U, as the strategies in (iv) are not weak limits of $\{0,1\}$ or of point masses, and $a - b \neq 0$ is continuous on R. Since U is connected and (i) holds for p_1 close to 1, each connected component of R has a boundary point in $U \setminus R$. There one of (i)–(iii) holds, and continuity gives $(a, b) \to (p_1, 1)$, $(p_1, b(p_1))$ or $(a(p_2), 1)$ respectively, with $a(p_2) < 1$ as $p_2 < \overline{p}$. Each limit has $a < b$, hence $a < b$ on R.

A.7 Proof of Proposition 3

Describe a signal of agent i by its high posterior $c_i \in [p_i, 1]$, where $c_i = p_i$ stands for $\{p_i\}$. For a fixed p_i, $\frac{\mathbb{P}(s_i=0)}{1-p_i} = \frac{c_i - p_i}{c_i(1-p_i)}$ is increasing in c_i, so it suffices to show that c_i is non-decreasing in p_j. By Lemma 1, the best reply to c_j is $c_i = \max\{r(c_j, \frac{p_j}{c_j}), p_i\}$, where $r(c, q)$ is the point at which N from (N) changes sign; it is continuous in (c, q). With $w = -\tilde{v}''$, which is odd and non-decreasing,

$$\partial_q N(s) = A(s) + B(s; c) = \int_0^s x\big[w(x) + w(c - x)\big]dx > 0,$$

since $w(x) + w(c - x)$ is at least $w(x) > 0$ if $x \leq c$ and equals $w(x) - w(x - c) \geq 0$ if $x > c$. Hence $r(c, q)$ is non-decreasing in q, and strictly increasing where $r(c, q) \in (0, 1)$, and $r(c, 1) = b(c)$ from Step 1 of the proof of Proposition 2.

Let $\rho(a) := r(a, \frac{p_1}{a})$, so that agent 2's best reply to a is $\max\{\rho(a), p_2\}$, and let $\Gamma(a) := \max\{r(c, \frac{p_2}{c}), p_1\}$ with $c = \max\{\rho(a), p_2\}$, for $a \in [p_1, 1]$. Γ is continuous and its fixed points are exactly the equilibria, so by Step 4 of the proof of Proposition 2 it has a unique fixed point a^*; since $\Gamma(p_1) \geq p_1$, $\Gamma(a) > a$ on $[p_1, a^*)$. As p_2 increases to p_2', $\Gamma(a)$ does not decrease for any a: if $\rho(a) \geq p_2'$, only the weight $\frac{p_2}{\rho(a)}$ increases; if $\rho(a) \leq p_2$, agent 2 pools and $b(p_2) \leq b(p_2')$; and if $p_2 < \rho(a) < p_2'$, then $r(\rho, \frac{p_2}{\rho}) \leq r(\rho, 1) = b(\rho) \leq b(p_2')$. Hence the fixed point does not decrease, and if $p_1 < a^* < 1$, $\Gamma(a^*)$ strictly increases, and so does the fixed point. As the argument does not use $p_2 \leq p_1$, it applies to both agents.

A.8 Proof of Corollary 1

Let $P_k(x) := x\big(1 - \frac{1}{\mathcal{E}_{\tilde{v}_k}(x)}\big)$ for $k \in \{e, n\}$, so that $P_k(a_k) = p$ with $a_k < 1$ by Theorem 1, and P_k is strictly increasing by its proof. Since $\mathcal{E}_{\tilde{v}_e}(x) > \mathcal{E}_{\tilde{v}_n}(x)$, $P_n(a_e) < P_e(a_e) = p = P_n(a_n)$, hence $a_e < a_n$.

A.9 Proof of Proposition 4

By Theorem 1 and Proposition 1, the equilibrium is $(\{0,a\};\{0,a\})$ with $a = a(p) > p$, where $a(p)$ is non-decreasing on $(0,1)$, as $a(p) = 1$ for $p \geq \overline{p}$. When both agents play $\{0,a\}$ with prior $p < a$, the firm gets

$$U(p,a) := \frac{p^2}{a} + \frac{2p(a-p)}{a}v(a) = p + p\Big(1 - \frac{p}{a}\Big)\big(2v(a) - 1\big).$$

For a fixed p, U is non-decreasing in a, as $1 - \frac{p}{a}$ and $2v(a) - 1$ are non-negative and non-decreasing. For a fixed a, $\partial_p U = 1 + \big(1 - \frac{2p}{a}\big)\big(2v(a) - 1\big) > 2\big(1 - v(a)\big) \geq 0$, as $1 - \frac{2p}{a} > -1$ and $0 < 2v(a) - 1 \leq 1$. Hence, for $p_1 < p_2$,

$$u_F(p_1) = U\big(p_1, a(p_1)\big) \leq U\big(p_1, a(p_2)\big) < U\big(p_2, a(p_2)\big) = u_F(p_2).$$

B. Proofs for Section 4

B.1 Equations for the noiseless rule

Fact (Boleslavsky and Cotton 2015, Lemma 3)**.** *The unique equilibrium for* $v(x) = \frac{1}{2} + \frac{1}{2}\cdot\operatorname{sign}(x)$ *has the form* (G,G)*, where for* $p \geq \frac{1}{2}$

$$G(x) = \begin{cases} \frac{x}{2p}, & x \leq 2 - 2p, \\ \frac{1}{p} - 1, & 2 - 2p \leq x < 1, \\ 1, & x = 1, \end{cases}$$

while for $p \leq \frac{1}{2}$

$$G(x) = \begin{cases} \frac{x}{2p}, & x \leq 2p, \\ 1, & x \in [2p, 1]. \end{cases}$$

Hence the firm's equilibrium utility is

$$u_F^0(p) = \mathbb{E}_G[\max(s_1, s_2)] = \begin{cases} \frac{4p}{3}, & 0 < p \le \frac{1}{2}, \\ 1 - \frac{(1-p)^2(4p-1)}{3p^2}, & \frac{1}{2} < p < 1. \end{cases}$$

B.2 Proof of Theorem 2

Lemma 6. *Let $v(x)$ be well S-shaped and $\hat{v}(x) := v(x) - \frac{1}{2}$. Then $\hat{v}(x) \le x(2-x)\hat{v}(1) \le x - \frac{x^2}{2}$ for all $x \in [0,1]$, and if $\hat{v}(x) = x - \frac{x^2}{2}$ for some $x \in (0,1)$, then $v = v^*$.*

Proof. For $0 \le t < s < 1$, s is a convex combination of t and 1 with weight $\frac{1-s}{1-t}$ on t, so the concavity of $v'(x)$ and $v'(1) \ge 0$ give $v'(s) \ge \frac{1-s}{1-t} v'(t)$. Hence $r(t) := \frac{v'(t)}{1-t}$ is non-decreasing on $[0,1)$: the density $v'(t)$ has a monotone likelihood ratio relative to the triangular density $1-t$. Let $T(x) := \int_0^x (1-t)dt = x - \frac{x^2}{2}$. For $x \in (0,1)$,

$$\frac{\hat{v}(x)}{T(x)} = \frac{\int_0^x r(t)(1-t)dt}{\int_0^x (1-t)dt} \le r(x) \le \frac{\int_x^1 r(t)(1-t)dt}{\int_x^1 (1-t)dt} = \frac{\hat{v}(1) - \hat{v}(x)}{T(1) - T(x)},$$

so $\hat{v}(x) \le \frac{T(x)}{T(1)}\hat{v}(1) = x(2-x)\hat{v}(1) \le x - \frac{x^2}{2}$, as $\hat{v}(1) \le \frac{1}{2}$. If $\hat{v}(x) = x - \frac{x^2}{2}$ for some $x \in (0,1)$, then $\hat{v}(1) = \frac{1}{2}$ and both inequalities in the display are equalities, so r, which is continuous on $[0,1)$ by property 1) in Appendix A.1, is constant there and equal to $2\hat{v}(1) = 1$. Hence $v'(t) = 1 - t$ on $[0,1]$, and $v = v^*$, as $v'(x)$ is even. □

By Theorem 1 and Proposition 1, the equilibrium is $(\{0,a\};\{0,a\})$, where $a = 1$ if and only if $p \ge \overline{p}$, and $p < a \le a^* := \min(2p, 1)$ by Lemma 4 and $\overline{p} \ge \frac{1}{2}$ (Lemma 3). As in the proof of Proposition 4, $u_F(p) = p + 2p\left(1 - \frac{p}{a}\right)\hat{v}(a)$, so, by Lemma 6,

$$u_F(p) \le p + 2p\left(1 - \frac{p}{a}\right)\left(a - \frac{a^2}{2}\right) \le p + 2p\left(1 - \frac{p}{a^*}\right)\left(a^* - \frac{a^{*2}}{2}\right) = u_F^{ws}(p),$$

where the second inequality holds as both factors are positive and strictly increasing in a on $(p, 1]$. This proves the bound, and equality holds if and only if $a = a^*$ and $\hat{v}(a) = a - \frac{a^2}{2}$.

If $p \geq \frac{1}{2}$, then $a^* = 1$, so equality means $a = 1$, i.e., $p \geq \overline{p}$, and $v(1) = 1$. For v^*, $\tilde{v}(x) = -x^2$ on $[0, 1]$, so $\overline{p} = \frac{1}{2}$ and $v^*(1) = 1$, and v^* attains the bound.

If $p < \frac{1}{2}$, then $a^* = 2p \in (0, 1)$, so equality requires $\hat{v}(2p) = 2p - 2p^2$, hence $v = v^*$ by Lemma 6. Conversely, for v^*, $\tilde{v}(x) = -x^2$ on $[0, 1]$, so $a = 2p$ by Theorem 1, and both inequalities hold with equality.

It remains to compare v^* with the noiseless rule. By the Fact in Appendix B.1, the utility of the firm under the noiseless rule is $\frac{4p}{3}$ for $p \leq \frac{1}{2}$ and $1 - \frac{(1-p)^2(4p-1)}{3p^2}$ for $p > \frac{1}{2}$. For $p \leq \frac{1}{2}$,

$$p + 2p^2(1-p) - \frac{4p}{3} = -\frac{p}{3}\left(6p^2 - 6p + 1\right) > 0 \Leftrightarrow p \in \left(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right).$$

For $p > \frac{1}{2}$,

$$2p - p^2 - 1 + \frac{(1-p)^2(4p-1)}{3p^2} = \frac{(1-p)^2(4p-1-3p^2)}{3p^2} = \frac{(1-p)^3(3p-1)}{3p^2} > 0.$$

B.3 Proof of Lemma 2

Since $v(-x) = 1 - v(x)$, for $s_1 \geq s_2$,

$$\begin{aligned} s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1) &= s_2 + (s_1 - s_2)v(s_1 - s_2) \leq s_2 + (s_1 - s_2)v(1) \\ &= v(1)\max(s_1, s_2) + (1 - v(1))\min(s_1, s_2) \leq \max(s_1, s_2), \end{aligned}$$

where the first inequality holds as $v(x)$ is increasing and $s_1 - s_2 \leq 1$, and the second as $v(1) \leq 1$, strictly if $s_1 \neq s_2$ and $v(1) < 1$. Hence, for any signals, the firm's utility

is at most $\mathbb{E}[\max(s_1, s_2)]$. For $s_2 \in [0,1]$, $\max(s, s_2)$ lies below its secant $s_2 + s(1 - s_2)$ on $[0,1]$, strictly on $(0,1)$ if $s_2 \in (0,1)$. Hence $\mathbb{E}_{G_1}[\max(s_1, s_2)] \leq s_2 + p(1 - s_2)$, with equality for all s_2 if G_1 is full revelation, and for $s_2 \in (0,1)$ only then. Averaging over s_2, $\mathbb{E}[\max(s_1, s_2)] \leq 2p - p^2$, with equality if and only if at least one agent plays full revelation. Moreover, $2p - p^2 = \mathbb{E}[\max(\theta_1, \theta_2)] = u_F^*(p)$.

So the firm's utility in every equilibrium is at most $u_F^*(p)$. Let (G_1^*, G_2^*) be an equilibrium with $\mathbb{E}[s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1)] = u_F^*(p)$. Then all inequalities above are equalities, so some agent plays full revelation, and since then $s_1 \neq s_2$ with positive probability, $v(1) = 1$. As all equilibria form a symmetric product set, $(\{0,1\}; \{0,1\})$ is then an equilibrium too. Conversely, if $(\{0,1\}; \{0,1\})$ is an equilibrium and $v(1) = 1$, then $s_1 - s_2 \in \{-1, 0, 1\}$ and both inequalities are equalities, so the firm attains $u_F^*(p)$.

B.4 Triangular selection rules

This subsection constructs the symmetric equilibrium used in Subsection 4.2 for the selection rules generated by i.i.d. $\varepsilon_i \sim U[0,u]$ with $u < 1$. Since $v(x)$ is piecewise quadratic, $v'(x)$ is piecewise linear, so the condition that $\int_0^1 v'(s_j - s_i) dG_i(s_i)$ be constant on $[a,b]$ reduces to a difference equation, and one can find $G_i(s_i)$ s.t. $\overline{v}(s_j)$ is linear on $[a,b]$.

Definition 5. Say χ is a proper u-partition of the interval $[a,b]$ if its CDF is given by:

$$G(x) = \begin{cases} 0, x < a \\ \frac{i+1}{k+1}, x \in [a + i \cdot u, b - (k-1-i) \cdot u), i \in \{0, \dots, k-1\} \\ \frac{i+1}{k}, x \in [b - (k-1-i) \cdot u, a + (i+1) \cdot u), i \in \{0, \dots, k-2\} \\ 1, x \geq b \end{cases}, \text{with } k = \lceil \tfrac{b-a}{u} \rceil$$

Remark 2. This G solves the functional equation $G(x - u) + G(x + u) = 2G(x)$, i.e. $\mathbb{P}(x < \chi \leq x + u) = \mathbb{P}(x - u < \chi \leq x)$, for all $x \in [a,b)$, subject to $\text{supp}(\chi) \subseteq [a,b]$.

Lemma 7. *Let $v(x)$ be the CDF of a triangular distribution on $[-u,u]$, $u \in (0,1)$. For a distribution K with finitely many support points in $[0,1]$, $S_K(x) := \int v'(x-t)\,dK(t)$ is continuous and piecewise linear, and outside a finite set*

$$S_K'(x) = \frac{1}{u^2}\Big[K\big((x,x+u)\big) - K\big((x-u,x)\big)\Big].$$

If K is the proper u-partition of $[a,b] \subseteq [0,1]$ and $k = \lceil \frac{b-a}{u} \rceil$, then K has mean $\frac{a+b}{2}$, $S_K = \frac{2ku-(b-a)}{u^2k(k+1)}$ on $[a,b]$, and S_K is non-increasing and convex on $[b,1]$.

Proof. As $v'(y) = \frac{(u-|y|)^+}{u^2}$ has slope $\frac{1}{u^2}$ on $(-u,0)$, $-\frac{1}{u^2}$ on $(0,u)$ and 0 for $|y| > u$, S_K is continuous and piecewise linear, with the stated derivative at every $x \notin \{t, t \pm u : K(\{t\}) > 0\}$. Let K be the proper u-partition of $[a,b]$. By the definition, its CDF jumps at $\ell_i := a + iu$ by $\frac{i+1}{k+1} - \frac{i}{k} = \frac{k-i}{k(k+1)}$ and at $r_i := b - iu$ by $\frac{k-i}{k(k+1)}$, $i = 0,\dots,k-1$, the two jumps adding up where $\ell_i = r_{i'}$; set the masses of ℓ_k and r_k to 0. As $\ell_i + r_i = a + b$ and ℓ_i, r_i have equal masses, the mean is $\frac{a+b}{2}$. For $x \in (a,b)$ outside the finite set, let $i := \lfloor \frac{x-a}{u} \rfloor \in \{0,\dots,k-1\}$ and $j := \lceil \frac{b-x}{u} \rceil \in \{1,\dots,k\}$. As each of $\{\ell_i\}$ and $\{r_i\}$ has spacing u, $(x-u,x)$ contains exactly the points ℓ_i and r_j of these lattices, and $(x,x+u)$ exactly ℓ_{i+1} and r_{j-1}, so $u^2k(k+1)S_K'(x) = (k-i-1) + (k-j+1) - (k-i) - (k-j) = 0$. So S_K is constant on $[a,b]$, equal to $S_K(a)$; the only atoms in $[a,a+u)$ are a, of mass $\frac{1}{k+1}$, and possibly $r_{k-1} \in (a,a+u]$, of mass $\frac{1}{k(k+1)}$, so $S_K(a) = \frac{ku+(u-r_{k-1}+a)}{u^2k(k+1)} = \frac{2ku-(b-a)}{u^2k(k+1)}$. For $x \geq b$, $S_K(x) = \int \frac{(u-(x-t))^+}{u^2}\,dK(t)$ with $x - t \geq 0$, a mixture of convex non-increasing functions of x. □

The proper u-partition of $[a,b]$ is the unique $G_i(s_i)$ supported in $[a,b]$ that makes $\overline{v}(s_j) = \int_0^1 v(s_j - s_i)dG_i(s_i)$ linear on $[a,b]$. Both agents playing it is often not an equilibrium, since $\overline{v}(s_j)$ beyond $[a,b]$ also matters, but it is the key ingredient: the next proposition constructs a symmetric equilibrium for all (u,p) from mixtures of proper u-partitions and, possibly, $\delta_{\{1\}}$.

Proposition 13. *Let $v(x)$ be the CDF of a triangular distribution on $[-u, u]$. Then a symmetric equilibrium signal strategy is:*

(i) If $p \leq \frac{1}{2}$, the signal is a proper u-partition of $[0, 2p]$.

(ii) If $p > \frac{1}{2}$ and $u \geq 2 - 2p$, the signal is $\{0, 1\}$.

(iii) Otherwise, the signal mixes a proper u-partition of $[0, a]$ for some a with a mass point at 1 and (possibly) with a proper u-partition of $[0, 1 - u]$.

B.5 Proof of Proposition 13

Full statement: let $v(x)$ be the CDF of a triangular distribution on $[-u, u]$; then there exists a symmetric equilibrium in which each agent's signal G is:

(i) Proper u-partition of $[0, 2p]$ if $p \leq \frac{1}{2}$

(ii) $\{0, 1\}$, if $p > \frac{1}{2}, u \geq 2 - 2p$.

(iii) $(\frac{2(1-p)}{2-a}, \frac{2p-a}{2-a})$ mix of the proper u-partition of $[0, a]$ with $\delta_{\{1\}}$, if (i) $p > \frac{1}{2}, u < 2-2p$, and (ii) $a \leq 1 - u$, where $a = 2ku - \frac{k(k+1)u^2}{2(1-p)}, k = \lceil \frac{2-2p}{u} \rceil - 1$.

The remaining case, in which the solution of the equations in (iii) has $a > 1 - u$, is constructed in Online Appendix OA.2.3; Proposition 5 does not use it.

If $u \geq 1$, then $v(x)$ is quadratic and $\tilde{v}(x) = -x^2$ on $[0, 1]$; hence I refer the reader to Proposition 1 and Theorem 1. Consider now $u < 1$. Then $v'(x) = \frac{\max(u-|x|,0)}{u^2}$, which is continuous, so $\overline{v}'$ is continuous; $v(x) = 0$ for $x < -u$, $v(x) = \frac{(u+x)^2}{2u^2}$ on $[-u, 0]$, and $v(x) = 1 - v(-x)$.

(i) Let $p \leq \frac{1}{2}$ and let χ be the proper u-partition of $[0, 2p]$, which has mean p by Lemma 7. Against χ, $\overline{v}' = S_\chi$ is constant on $[0, 2p]$ and non-increasing on $[2p, 1]$ by Lemma 7, so $\overline{v}$ is concave on $[0, 1]$ and affine on $[0, 2p] \ni p$. Hence every feasible signal

supported on $[0, 2p]$, χ included, is a best reply, and a pair of proper u-partitions of $[0, 2p]$ is an equilibrium.

(ii) Suppose that $p > \frac{1}{2}$, $2 - 2p \le u < 1$; then $\overline{v}(x) = (1-p)v(x) - pv(1-x)$. Then

$$\overline{v}'(x) = \frac{(1-p)\max(u-x, 0) + p\max(u+x-1, 0)}{u^2},$$

where the numerator is a piecewise linear convex function, hence $\overline{v}(x)$ is concave and then convex, and we need to check that $\overline{v}'(0) \le \overline{v}(1) - \overline{v}(0)$.

$\overline{v}'(0) = \frac{1-p}{u}$, $\overline{v}(1) - \overline{v}(0) = v(1) - v(0) = \frac{1}{2}$, so $\{0, 1\}$ is a best reply iff $2 - 2p \le u$.

(iii) Suppose $p > \frac{1}{2}, u < 2 - 2p$, and $a \le 1 - u$, where $a = 2ku - \frac{k(k+1)u^2}{2(1-p)}$ for $k = \lceil \frac{2-2p}{u} \rceil - 1$. Let $\omega := \frac{2(1-p)}{2-a}$ and let G be the $(\omega, 1-\omega)$ mix of the proper u-partition χ of $[0, a]$ with $\delta_{\{1\}}$; it has mean $\omega\frac{a}{2} + 1 - \omega = p$ by Lemma 7.

First, $\lceil \frac{2-2p}{u} \rceil - 1 = \lceil \frac{a}{u} \rceil$, so k is the same in the proper u-partition of $[0, a]$:

$$\left\lceil \frac{a}{u} \right\rceil = \left\lceil 2k - \frac{k(k+1)u}{2(1-p)} \right\rceil = k,$$

since $k < \frac{2(1-p)}{u} \le k + 1$, hence $k \le \frac{k(k+1)u}{2(1-p)} < k + 1$.

Against G, $\overline{v}'(x) = \omega S_\chi(x) + (1-\omega)\frac{(u+x-1)^+}{u^2}$, whose second term vanishes on $[0, a]$, as $a \le 1 - u$. So, by Lemma 7, $\overline{v}$ is affine on $[0, a]$ with slope $\overline{v}'(0) = \omega\frac{2ku-a}{u^2k(k+1)}$. Since $v(1-s) = 1$ for $s \le a \le 1 - u$,

$$\overline{v}(1) - \overline{v}(0) = \omega\mathbb{E}[v(\chi)] + \frac{1-\omega}{2}, \qquad \mathbb{E}[v(\chi)] = 1 - \frac{1}{2(k+1)} - \frac{(ku-a)^2}{2u^2k(k+1)},$$

and, after cancelling the factor $2-a$, the equation $\overline{v}(1) - \overline{v}(0) = \overline{v}'(0)$ reduces to $2(1-p)a = 4(1-p)ku - k(k+1)u^2$, whose root is $a = 2ku - \frac{k(k+1)u^2}{2(1-p)}$. Hence $\int_a^1 (\overline{v}'(x) - \overline{v}'(a))dx = 0$.

On $[a, 1]$ both summands of $\overline{v}'(x)$ are convex: the first by Lemma 7, the second as a maximum of linear functions. Hence $\overline{v}'(x) - \overline{v}'(a)$ is convex on $[a, 1]$, vanishes at

a and integrates to zero, so $\exists c : \overline{v}'(x) \leq \overline{v}'(a)\ \forall x \in [a, c]$ and $\overline{v}'(x) \geq \overline{v}'(a)\ \forall x \in [c, 1]$. Therefore $\overline{v}(x) \leq \overline{v}(a) + (x - a)\overline{v}'(a)$ on $[a, 1]$, with equality at $x = 1$, so $[0, a] \cup \{1\}$ is the concavification support, and (G, G) is an equilibrium.

Signals not ranked in the Blackwell order. For the example and the claim after Proposition 5, let $p \geq \frac{5}{9}$ and $u \in [1-p, 2-2p)$. Then $k = 1$ in (iii), and $a = 2u - \frac{u^2}{1-p} \leq 1-u$, as $3u - \frac{u^2}{1-p}$ is maximized at $u = \frac{3(1-p)}{2}$, where it equals $\frac{9(1-p)}{4} \leq 1$. The proper u-partition of $[0, a]$ is $\{0, a\}$ with equal probabilities, so G_u puts probabilities $\frac{1-p}{2-a}$, $\frac{1-p}{2-a}$ and $\frac{2p-a}{2-a}$ on $\{0, a, 1\}$. As $\frac{da}{du} = 2 - \frac{2u}{1-p} < 0$ for $u > 1 - p$, a higher u lowers the probability of 0 and raises that of 1, so no two of these signals are ranked in the Blackwell order.

B.6 Proof of Proposition 5

Throughout, G_u is the equilibrium signal constructed in Proposition 13, and proper u-partitions are as defined in Appendix B.4.

(i) Let $p \leq \frac{1}{2}$, $u < 2p$, $k = \lceil \frac{2p}{u} \rceil \geq 2$ and $r = 2p - (k-1)u \in (0, u]$. The proper u-partition of $[0, 2p]$ has atoms iu with masses $\frac{k-i}{k(k+1)}$ and atoms $iu + r = 2p - (k-1-i)u$ with masses $\frac{i+1}{k(k+1)}$, $i = 0, \ldots, k-1$. For fixed k, as u increases, iu moves at rate i and $iu + r$ at rate $-(k-1-i)$, so $I_{G_u}(t) := \int_0^t G_u(x)dx$ satisfies $\partial_u I_{G_u}(t) = -M(t)$, where $M(t)$ sums rate times mass over the atoms below t. The pair $\{iu, iu + r\}$ contributes $\frac{2i+1-k}{k(k+1)}$ and the atom iu alone $\frac{i(k-i)}{k(k+1)}$, hence $M(t) = 0$ for $t \in (ju, ju + r]$ and $M(t) = \frac{(j+1)(j+1-k)}{k(k+1)} < 0$ for $t \in (ju + r, (j+1)u]$, $j \leq k-2$. So $I_{G_u}(t)$ is non-decreasing in u, and strictly increasing for t in a set of positive measure, i.e. G_u is a strict mean-preserving spread of $G_{u'}$ for $u' < u$ with the same k. At $u = \frac{2p}{k}$, $r = u$ and G_u is the uniform distribution on $\{0, \frac{2p}{k}, \ldots, 2p\}$, which is also its limit as $u \uparrow \frac{2p}{k}$, where $k+1$ is used and $r \to 0$; so G_u is continuous in u and the ranking extends to $(0, 2p]$. By Proposition 13, $G_u = \{0, \min(2p, 1)\}$ if $u \geq 2p$ and $p \leq \frac{1}{2}$, or $u \geq 2 - 2p$ and $p > \frac{1}{2}$.

(ii) Since $v(-x) = 1 - v(x)$, with $d = s_1 - s_2$,

$$u_F = \mathbb{E}[s_1 v(d) + s_2 v(-d)] = \mathbb{E}[\max(s_1, s_2)] - \mathbb{E}\big[|d|\big(1 - v(|d|)\big)\big],$$

and $v(x) = 1 - \frac{(u-x)^2}{2u^2}$ for $x \in [0, u]$. Let $p \leq \frac{1}{2}$. For $u \geq 2p$, $G_u = \{0, 2p\}$ and

$$u_F = \frac{p}{2} + p\, v(2p) = \frac{3p}{2} - \frac{p(u-2p)^2}{2u^2},$$

which equals $\frac{3p}{2}$ at $u = 2p$ and is strictly decreasing on $[2p, 1]$. For $u < 2p$, $u_F \leq \mathbb{E}[\max(s_1, s_2)] < \frac{3p}{2}$ by the chord argument of the proof of Lemma 2 on $[0, 2p]$, which gives $\mathbb{E}_{s_1}[\max(s_1, s_2)] \leq p + \frac{s_2}{2}$, strictly for $s_2 \in (0, 2p)$, which G_u charges.

Moreover, for $u = 2p$, no equilibrium gives the firm more than $\frac{3p}{2}$. As equilibria form a product set and $(\{0, 2p\}; \{0, 2p\})$ is one, every equilibrium signal is a best reply to $\{0, 2p\}$, against which $\overline{v}(s) = \frac{1}{2}v(s) + \frac{1}{2}v(s - 2p)$ lies below the line ℓ through $\big(0, \overline{v}(0)\big)$ and $\big(2p, \overline{v}(2p)\big)$. On $[2p, 1]$, $\overline{v}$ is concave, as $v(s) = 1$ there, and strictly concave on $(2p, \min(4p, 1))$, so it touches ℓ only at $2p$. Hence every equilibrium signal is supported on $[0, 2p]$, and $u_F \leq \mathbb{E}[\max(s_1, s_2)] \leq \frac{3p}{2}$ by the chord argument above.

Let $p > \frac{1}{2}$. For $u \in [2-2p, 1]$, $G_u = \{0, 1\}$ and $v(1) = 1$, so $u_F = u_F^*(p)$. For $u < 2-2p$, $v(x)$ does not implement a fully revealing equilibrium by the proof of Proposition 13 (ii), so $u_F < u_F^*(p)$ by Lemma 2.

(iii) The proper u-partition of $[0, a]$ with $k = \lceil \frac{a}{u} \rceil$ puts mass $\frac{1}{k}$ on each pair $\{iu, iu + r\} \subset [iu, (i+1)u]$, $i = 0, \ldots, k-1$, so it converges weakly to $U[0, a]$ as $u \to 0$. If $p \leq \frac{1}{2}$, G_u is the proper u-partition of $[0, 2p]$ and converges to $U[0, 2p]$. If $p > \frac{1}{2}$ and $u < \min(2p - 1, 2 - 2p)$, then with k from Proposition 13 (iii), we have $a \in ((k-1)u, ku]$ and $ku < 2 - 2p \leq (k+1)u$, so $a \in (2 - 2p - 2u, 2 - 2p)$ and $a < 1 - u$. Hence case (iii) applies, $a \to 2 - 2p$, the weight $\frac{2(1-p)}{2-a} \to \frac{1-p}{p}$, and G_u converges to the $(\frac{1-p}{p}, \frac{2p-1}{p})$ mix of $U[0, 2 - 2p]$ and $\delta_{\{1\}}$ — exactly the distributions from Appendix B.1.

B.7 Robustness across unequal priors

Lemma 8. *Let $S \subseteq (0,1)$. A selection rule $v(x)$ implements a fully revealing equilibrium for all $(p_1, p_2) \in S^2$ if and only if it does so for all (p, p) with $p \in S$.*

Proof. Let agent j with prior p_j play full revelation. Then $\overline{v}(s_i) - \frac{1}{2} = (1-p_j)\hat{v}(s_i) - p_j\hat{v}(1-s_i)$ does not depend on p_i, and full revelation is a best reply of agent i for any $p_i \in (0,1)$ if and only if $\overline{v}$ lies below its secant from 0 to 1:

$$(1-p_j)\hat{v}(x) - p_j\hat{v}(1-x) \leq \hat{v}(1)(x-p_j) \quad \forall x \in (0,1).$$

Hence $(\{0,1\};\{0,1\})$ is an equilibrium for (p_1, p_2) if and only if this condition holds for $p_j \in \{p_1, p_2\}$, so it is an equilibrium for all $(p_1, p_2) \in S^2$ if and only if the condition holds for all $p_j \in S$, i.e., if and only if it is an equilibrium for all (p, p), $p \in S$.

B.8 Proof of Proposition 6

Let $a := \hat{v}(1)$; since $v(x) \leq 1$ is increasing, $a \in [0, \frac{1}{2}]$. By the proof of Lemma 8, $v(x)$ implements a fully revealing equilibrium for (p, p) if and only if $(1-p)\hat{v}(x) - p\hat{v}(1-x) \leq a(x-p)$ for all $x \in (0,1)$. Applying it also at $1-x$ and summing the two inequalities with weights $1-p$ and p gives $(1-2p)\big(\hat{v}(x) - ax\big) \leq 0$. Taking $p = p_l$ and $p = p_h$ yields $\hat{v}(x) = ax$, which proves (ii); then the condition holds with equality for all $p \in (0,1)$, and Lemma 8 extends full revelation to all pairs of priors, which proves (i).

B.9 Proof of Proposition 7

(i) With a and g as in Definition 3, $\hat{v}(x) = ax + g(x)$ on $[0,1]$ and $g(0) = g(1) = 0$, so the condition $(1-p)\hat{v}(x) - p\hat{v}(1-x) \leq a(x-p)$ for all $x \in (0,1)$ from the proof of Lemma 8 reads $(1-p)g(x) - pg(1-x) \leq 0$ for all $x \in (0,1)$. It is linear in p, so it holds for all $p \in [\frac{1}{2}, 1)$ if and only if it holds at $p = \frac{1}{2}$ and $p = 1$. At $p = 1$ it reads $g \geq 0$ on $(0,1)$; at

$p = \frac{1}{2}$ it reads $g(x) \le g(1-x)$, which, applied at x and $1-x$, means $g(x) = g(1-x)$. The case $p \in (0, \frac{1}{2}]$ is the same with $p = 0$, where the condition reads $g \le 0$. For a balanced $v(x)$ and $x < y$ in $[0,1]$, $\hat{v}(x) \le \hat{v}(y)$ and $\hat{v}(1-y) \le \hat{v}(1-x)$ read $g(x) - g(y) \le a(y-x)$ and $g(y) - g(x) \le a(y-x)$, so $\hat{v}$ is increasing if and only if $|g(x) - g(y)| \le a|x-y|$.

(ii) As all equilibria form a product set, it suffices that full revelation is each agent's unique best reply to full revelation, i.e., that the condition above is strict on $(0,1)$. For a balanced $v(x)$, the condition reads $(1-2p)g(x) < 0$, which holds for all $p > \frac{1}{2}$ [$p < \frac{1}{2}$] if and only if $g > 0$ [$g < 0$] on $(0,1)$.

B.10 Proof of Corollary 2

Suppose that agent j plays $\{0,1\}$. Then $\overline{v}(s_i) = (1-p_j)v(s_i) + p_j v(s_i - 1)$, and $\{0,1\}$ is a best reply of agent i only if the secant from 0 to 1 lies weakly above $\overline{v}(s_i)$ on $[0,1]$, which requires $\overline{v}'(0) \le \overline{v}(1) - \overline{v}(0) \le \overline{v}'(1)$. Since $v'(x)$ is even, $\overline{v}'(0) = (1-p_j)v'(0) + p_j v'(1)$ and $\overline{v}'(1) = (1-p_j)v'(1) + p_j v'(0)$, so this requires $(2p_j - 1)\big(v'(0) - v'(1)\big) \ge 0$, i.e., $p_j \ge \frac{1}{2}$. Hence in a fully revealing equilibrium both priors are at least $\frac{1}{2}$.

If $v(x)$ is generated by i.i.d. shocks with a square-integrable density f, then $v'(x) = \int f(t)f(t+x)dt$ is continuous, and by the Cauchy–Schwarz inequality,

$$v'(1) = \int f(t)f(t+1)dt \le \int f^2(t)dt = v'(0),$$

with equality only if $f(t+1) = f(t)$ almost everywhere, which is impossible for a probability density. Hence $v'(0) > v'(1)$.

B.11 First best with priors on both sides of one half

Let $p_l := \min(p_1, p_2) < \frac{1}{2} < p_h := \max(p_1, p_2)$, and call the agents l and h accordingly.

The rule. Let $\gamma(x) := \varepsilon x(1-2x)$, so that $v(x) = \frac{1}{2} + x + \gamma(x)$ on $[0, \frac{1}{2}]$ and $v = 1$ on $[\frac{1}{2}, 1]$. Against $\{0,1\}$ played by agent l, agent h's value function is $\overline{v}_h(s) = (1-p_l)\big(\frac{1}{2} + s + \gamma(s)\big)$

on $[0, \frac{1}{2}]$ and $(1-p_l) + p_l\big(s - \frac{1}{2} - \gamma(1-s)\big)$ on $[\frac{1}{2}, 1]$. With $\ell(s) := (1-p_l) + p_l(s - \frac{1}{2})$, $\ell - \overline{v}_h = p_l\gamma(1-s) > 0$ on $(\frac{1}{2}, 1)$ and $\ell - \overline{v}_h = (\frac{1}{2} - s)\big[(1-2p_l) - 2\varepsilon s(1-p_l)\big]$ on $[0, \frac{1}{2})$, which is positive if $\varepsilon(1-p_l) \leq 1 - 2p_l$. Then $\{\frac{1}{2}, 1\}$ is agent h's unique best reply, with weight $\mu := 2p_h - 1$ on 1. Against it, the chord of agent l's value function from 0 to 1 lies strictly above it on $(0,1)$ if $(1-2\mu)\varepsilon \leq \frac{\mu}{2}$, i.e. $\varepsilon(3-4p_h) \leq p_h - \frac{1}{2}$, so $\{0,1\}$ is agent l's unique best reply. As the equilibria form a product set, $(\{\frac{1}{2}, 1\}; \{0,1\})$, up to the labels, is the unique equilibrium, and as $v(\frac{1}{2}) = 1$, the firm selects agent h unless agent l is revealed good and agent h is not, which gives $u_F = p_l + (1-p_l)p_h$, the first best.

Necessity. Let an equilibrium attain the first best. As $u_F \leq \mathbb{E}[\max(s_1, s_2)] \leq p_1 + p_2 - p_1p_2$, with the second inequality strict unless some agent j reveals fully, agent j plays $\{0,1\}$, and the first inequality is an equality only if $v(1) = 1$ and $v(x) = v(1-x) = 1$ for every $x \in (0,1)$ in the support of the other agent's signal. If both agents reveal fully, their conditions are those of Proposition 6 at p_l and p_h, so v is linear. Otherwise, let $x_0 := \min\{x : v(x) = 1\} \leq \frac{1}{2}$. Against $\{0,1\}$, the other agent's value function $(1-p_j)v(s) + p_j\big(1 - v(1-s)\big)$ equals $1 - p_j$ on $[x_0, 1-x_0]$ and exceeds it at 1, so a line supporting it on that interval touches it only at x_0, and the signal is supported on $\{0, x_0, 1\}$; the secant conditions of agent j against such signals hold only if this agent is h, $x_0 \geq 1 - p_h$ and his signal is $\{x_0, 1\}$. For $s < x_0$, $\overline{v}_h(s) = (1-p_l)v(s)$ lies below the line through $(x_0, 1-p_l)$ and $(1, 1-\frac{p_l}{2})$, so $\liminf_{s\uparrow x_0} \frac{v(x_0)-v(s)}{x_0-s} \geq \frac{p_l}{2(1-p_l)(1-x_0)} > 0$, whereas $v = 1$ to the right of x_0. So v is not differentiable at x_0. A rule generated by i.i.d. shocks with a square-integrable density is continuously differentiable, and by Corollary 2 it is not linear, so it attains the first best in no equilibrium.

C. Proofs for Subsection 4.4 and Section 5

C.1 Censorship under a general prior

For an arbitrary prior F, let $F^{-1}(u) := \inf\{x \in [0,1] : F(x) \geq u\}$ for $u \in (0,1]$, $F^{-1}(0) := 0$, and let $F^{-1}(u^+)$ be its right limit, with $F^{-1}(1^+) := F^{-1}(1)$. For $q \in [0,1)$ let $m(q) := \frac{1}{1-q}\int_q^1 F^{-1}(u)du$, and $m(1) := \overline{\theta}$. With $U \sim U[0,1]$, the quantile censorship G_q reveals the type $F^{-1}(U)$ if $U < q$ and sends $m(q)$ otherwise:

$$G_q(x) = \begin{cases} \min\{F(x), q\}, & x < m(q), \\ 1, & x \geq m(q). \end{cases}$$

The next lemma extends the optimality of upper censorship to an arbitrary prior; Appendix C.3 proves it after the proof of Lemma 10, whose steps it uses.

Lemma 9. *Let $v(x)$ be well S-shaped and F any prior on $[0,1]$. Against every rival signal K, some quantile censorship G_q, $q \in [0,1]$, is a best reply.*

The existence and uniqueness in Proposition 9 follow from the next lemma; the bound $q \leq \frac{1}{2}$ and the utility bound are proved in Appendix C.6.

Lemma 10. *Let $v(x)$ be well S-shaped and F any prior.*

(i) There are $q^* \in (0,1]$ and $\theta^* \in [F^{-1}(q^*), F^{-1}(q^{*+})]$ such that (G_{q^*}, G_{q^*}) is an equilibrium and either $\theta^* < m^* := m(q^*)$ and

$$\int_0^{q^*} \Big[\tilde{v}(m^* - F^{-1}(u)) - \tilde{v}(\theta^* - F^{-1}(u)) - (m^* - \theta^*)\tilde{v}'(m^* - F^{-1}(u))\Big]du + \tilde{v}(m^* - \theta^*)(1 - q^*) = 0, \tag{QC}$$

or $q^* = 1$, so that both agents reveal fully, and $\int_0^1 v'(x-t)\,dF(t)$ is nondecreasing on $[\underline{\theta}, \overline{\theta}]$.

(ii) Every q and $\theta \in [F^{-1}(q), F^{-1}(q^+)]$ with $\theta < m(q)$ that solve (QC) *give an equilibrium* (G_q, G_q).

(iii) If $v'(x)$ is not linear near 0, the equilibrium is unique; all solutions in (ii) then have the same q and induce the same signal.

C.2 Proof of Lemma 10

Preliminaries. By property 1) in Appendix A.1, v is C^1 on $[-1,1]$ with a concave derivative, but in general it is not C^2 (e.g. $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$). For a distribution K on $[0,1]$, let $w_K(x) := \int_0^1 \tilde{v}(x-y)dK(y)$. Since $v(x) = \frac{1}{2} + v'(0)x + |v(1) - \frac{1}{2} - v'(0)|\tilde{v}(x)$ and every feasible signal has mean $\mathbb{E}[\theta] =: \mu$, w_K has the same best replies as $\overline{v}(\cdot, K)$. It is C^1 on $[0,1]$, and $w_K'(x) = \int_0^1 \tilde{v}'(x-y)dK(y)$ is concave.

Quantile facts: F^{-1} is non-decreasing and left-continuous, and $F^{-1}(u) \le x \Leftrightarrow u \le F(x)$. Let $\overline{q} := F(\overline{\theta}-)$, so that $F^{-1} < \overline{\theta}$ on $(0, \overline{q})$ and $F^{-1} = \overline{\theta}$ on $(\overline{q}, 1]$. As F^{-1} is integrable, m is continuous on $[0,1)$, and $F^{-1}(q^+) \le m(q) \le \overline{\theta}$ gives $m(q) \to \overline{\theta} = m(1)$ as $q \to 1$. For $q < \overline{q}$, F^{-1} is not a.e. constant on $(q,1)$, so $m(q) > F^{-1}(q^+)$; for $q \ge \overline{q}$, $m(q) = \overline{\theta}$ and $G_q = F$. G_q is the distribution of $\mathbb{E}[F^{-1}(U)|\, U\mathbf{1}\{U < q\}]$, so it is feasible.

The curve. Let $\Gamma := \{(q,\theta) : q \in [0,\overline{q}],\ F^{-1}(q) \le \theta \le F^{-1}(q^+)\}$, with $F^{-1}(\overline{q}^+) := \overline{\theta}$, i.e. the graph of F^{-1} on $[0,\overline{q}]$ with its jumps filled in. As $\phi(q) := q + F^{-1}(q)$ is strictly increasing, for each $r \in [0, \overline{q} + \overline{\theta}]$ there is a unique $q(r)$ with $\phi(q(r)) \le r \le \phi(q(r)^+)$, and $q(r)$ is 1-Lipschitz, since $q(r) > q(r')$ gives $r - r' \ge q(r) - q(r') + F^{-1}(q(r)) - F^{-1}(q(r')^+) \ge q(r) - q(r')$. So $r \mapsto (q(r), r - q(r))$ is a continuous path that traverses Γ from $(0,0)$ to $(\overline{q}, \overline{\theta})$. By the quantile facts, $\theta < m(q)$ on $\Gamma \setminus \{(\overline{q}, \overline{\theta})\}$.

For $(q,\theta) \in \Gamma$, let $w_q := w_{G_q}$, let L_q be the tangent to w_q at $m(q)$, and let $E(q,\theta) := L_q(\theta) - w_q(\theta)$. Since $\tilde{v}$ is odd with $\tilde{v}'(0) = 0$, $E(q,\theta)$ is the left-hand side of (QC), and

$$E(q,\theta) = \int_\theta^{m(q)} \big(w_q'(z) - w_q'(m(q))\big)dz.$$

The function E is continuous on Γ, as m is continuous and $\tilde{v}, \tilde{v}'$ are both uniformly continuous, and $E(\overline{q}, \overline{\theta}) = 0$.

Step 1 (verification). Let K be any distribution on $[0,1]$, $w := w_K$, $(q,\theta) \in \Gamma$ with $\theta < m := m(q)$, and L the tangent to w at m with $L(\theta) = w(\theta)$. Then G_q is a best reply to K. Indeed, let $D := L - w$, so that $D(\theta) = D(m) = D'(m) = 0$ and $D' = w'(m) - w'$. If $w'(x) > w'(m)$ for some $x > m$, then $w' < w'(m)$ on $[0,m)$ by concavity, so D is strictly increasing on $[0,m]$, contradicting $D(\theta) = D(m)$. Hence D is non-decreasing, and non-negative, on $[m,1]$. The set of $x \leq m$ with $w'(x) \geq w'(m)$ is an interval $[\alpha, m]$, so D is non-decreasing on $[\theta, \alpha]$ and non-increasing on $[\max(\alpha,\theta), m]$, and $D(\theta) = D(m) = 0$ gives $D \geq 0$ on $[\theta, m]$. If $\alpha > \theta$, then $w'(\theta) < w'(\alpha)$, so w' is non-decreasing on $[0,\theta]$ and $w'(\theta) < w'(m)$. If $\alpha \leq \theta$, then $D = 0$ on $[\theta, m]$, so w' is constant on $[\theta, m]$, hence maximal there and non-decreasing on $[0,\theta]$. In both cases

$$\pi(x) := \begin{cases} w(x), & x \leq \theta, \\ L(x), & x \geq \theta, \end{cases}$$

is convex, $\pi \geq w$, and $\pi = w$ on $[0,\theta] \cup \{m\}$. For $u < q$, $F^{-1}(u) \leq F^{-1}(q) \leq \theta$, and for $u > q$, $F^{-1}(u) \geq F^{-1}(q^+) \geq \theta$, where π is affine. Hence $\operatorname{supp}(G_q) \subseteq [0,\theta] \cup \{m\}$, and

$$\int \pi \, dF = \int_0^q \pi(F^{-1}(u)) du + (1-q)\pi(m) = \int \pi \, dG_q = \int w \, dG_q.$$

So every mean-preserving contraction G' of F satisfies $\int w \, dG' \leq \int \pi \, dG' \leq \int \pi \, dF = \int w \, dG_q$.

Step 2 (full revelation). If w_K' is non-decreasing on $[\underline{\theta}, \overline{\theta}]$, then F is a best reply to K: w_K is convex on $[\underline{\theta}, \overline{\theta}]$, which contains the support of every mean-preserving contraction G' of F, so $\int w_K \, dG' \leq \int w_K \, dF$.

Step 3 (existence). Let $\underline{\theta} < \overline{\theta}$. At the start of the path, $E(0,\theta) = \tilde{v}(\mu - \theta) < 0$ for

$\theta \in [0, \underline{\theta}]$, as $\theta \leq \underline{\theta} < \mu$. Let $g := w'_F$, which is concave.

If g is non-decreasing on $[\underline{\theta}, \overline{\theta}]$, then $F = G_1$ is a best reply to F by Step 2, so (G_1, G_1) is an equilibrium, which is the second alternative in (i).

Otherwise, there are $\underline{\theta} \leq x_1 < x_2 < \overline{\theta}$ with $g(x_1) - g(x_2) =: 2\gamma > 0$ (if $x_2 = \overline{\theta}$, move x_2 slightly to the left, by continuity). Let $Y_q := F^{-1}(U)\mathbf{1}\{U < q\} + m(q)\mathbf{1}\{U \geq q\} \sim G_q$. As $q \to \overline{q}$, $Y_q \to Y_{\overline{q}}$ in probability, since they differ only on $\{q \leq U < \overline{q}\}$ and by $|m(q) - m(\overline{q})|$. Since $G_{\overline{q}} = F$ and $\tilde{v}'$ is uniformly continuous, $w'_q \to g$ uniformly. Hence there is a neighborhood N of $(\overline{q}, \overline{\theta})$ in Γ on which $w'_q(x_1) > w'_q(x_2)$ and $\theta > x_2$. For $(q, \theta) \in N$, the concave w'_q is then strictly decreasing on $[x_2, 1] \supseteq [\theta, m(q)]$, so $E(q, \theta) > 0$ whenever $\theta < m(q)$. By the intermediate value theorem along the path, E has a zero $(q^*, \theta^*) \in \Gamma \setminus \{(\overline{q}, \overline{\theta})\}$, where $\theta^* < m(q^*)$, and $q^* > 0$ since $E < 0$ for $q = 0$. By Step 1 with $K = G_{q^*}$, (G_{q^*}, G_{q^*}) is an equilibrium. This proves (i), and Step 1 applied with $K = G_q$ proves (ii).

Step 4 (uniqueness). Now suppose that $v'(x)$ is not linear near 0. First, let (q, θ) be as in (ii), so $q < 1$, and let $w := w_q$, $m := m(q)$, with D, α, π as in Step 1 for $K = G_q$. By property 4) in Appendix A.1, w' is affine on no interval that G_q charges, and $G_q(\{m\}) \geq 1 - q > 0$. Hence $\alpha > \theta$, as otherwise w' is constant on $[\theta, m]$. Also $D > 0$ on $(\theta, 1] \setminus \{m\}$: a zero of D in $(m, 1]$, or in $[\alpha, m)$, makes w' constant on an interval with endpoint m, and a zero in (θ, α) would give $D = 0$ on $[\theta, x]$, i.e. $w' = w'(m)$ there, while $w' < w'(m)$ on $[0, \alpha)$. Finally, w' is strictly increasing on $[0, \theta]$: it is non-decreasing there and concave, so if it were constant on a subinterval of $[0, \theta]$, it would attain its maximum there, but $w'(\theta) < w'(m)$.

Let G' be a best reply to G_q, so that both inequalities in Step 1 are equalities. The first one gives $\text{supp}(G') \subseteq [0, \theta] \cup \{m\}$. Writing $\pi(x) = \pi(0) + \pi'(0)x + \int_0^1 (x - y)^+ d\pi'(y)$ and $I_K(y) := \int_0^y K(x)dx$, the second one gives $\int_0^1 \big(I_F(y) - I_{G'}(y)\big)d\pi'(y) = 0$, where $I_{G'} \leq I_F$. As $d\pi'$ charges every subinterval of $[0, \theta]$, $I_{G'} = I_F$ on $[0, \theta]$, so $G' = F$ on $[0, \theta)$. The

rest of G', of mass $1 - F(\theta-)$, lies on $\{\theta, m\}$, and its first moment is $\int_{[\theta,1]} x\, dF$. As G_q also equals F on $[0, \theta)$ and puts $q - F(\theta-)$ on θ and $1 - q$ on m, these two moment conditions, with $\theta \neq m$, give $G' = G_q$. So G_q is the unique best reply to itself.

Second, if $g = w'_F$ is non-decreasing on $[\underline{\theta}, \overline{\theta}]$, it is strictly increasing there: a concave non-decreasing g that is constant on a subinterval is constant up to $\overline{\theta}$, i.e. affine on an interval that F charges, contradicting property 4). So w_F is strictly convex on $[\underline{\theta}, \overline{\theta}]$, and the same argument gives $G' = F$ for every best reply G' to F.

So in both cases of (i), the equilibrium signal is the unique best reply to itself. As all equilibria form a product set, every equilibrium strategy is a best reply to it, so the equilibrium is unique, and every solution in (ii) induces the same signal. A signal G_q with $\theta < m(q)$ determines q, as its top atom is at $m(q)$ with mass $1 - q$, and the revealed types lie below $m(q)$. So q is unique, and θ can vary only within $[F^{-1}(q), F^{-1}(q^+)]$, that is, within a gap in $\text{supp}(F)$.

C.3 Proof of Lemma 9

Steps 1–3 refer to the proof of Lemma 10, whose notation I use.

Let $\underline{\theta} < \overline{\theta}$, $w := w_K$, let L_x be the tangent to w at x, and let $e(q, \theta) := L_{m(q)}(\theta) - w(\theta) = \int_\theta^{m(q)} (w'(z) - w'(m(q)))dz$, which is continuous on Γ.

(a) If $e(0, \underline{\theta}) \geq 0$, then pooling $G_0 = \{\mu\}$ is a best reply. Let $D := L_\mu - w$, so $D(\underline{\theta}) \geq 0 = D(\mu) = D'(\mu)$. If $w'(x) > w'(\mu)$ for some $x > \mu$, then $w' < w'(\mu)$ on $[0, \mu)$, so $D(\underline{\theta}) < D(\mu) = 0$, a contradiction. So $D \geq 0$ on $[\mu, 1]$. On $[\underline{\theta}, \mu]$, D first increases and then decreases, so $D \geq \min(D(\underline{\theta}), D(\mu)) \geq 0$. Hence $L_\mu \geq w$ on $[\underline{\theta}, \overline{\theta}]$, and $\int w\, dG' \leq L_\mu(\mu) = w(\mu)$ for every mean-preserving contraction G' of F.

(b) If w' is non-decreasing on $[\underline{\theta}, \overline{\theta}]$, then $F = G_1$ is a best reply, by Step 2.

(c) Otherwise, $e(0, \underline{\theta}) < 0$, and as in Step 3 there are $x_1 < x_2 < \overline{\theta}$ with $w'(x_1) > w'(x_2)$. So w' is strictly decreasing on $[x_2, 1]$, and $e(q, \theta) > 0$ for $(q, \theta) \in \Gamma$ with $x_2 < \theta < m(q)$,

which holds near $(\overline{q}, \overline{\theta})$. Since the path from $(0, \underline{\theta})$ to $(\overline{q}, \overline{\theta})$ is continuous, e has a zero (q, θ) with $\theta < m(q)$, and G_q is a best reply by Step 1.

C.4 Median censorship

This subsection shows that, for $v(x) = \frac{1}{2} + x - \frac{1}{2}x|x|$ and every prior, $(G_{1/2}, G_{1/2})$ is an equilibrium, and the unique one, and $q = \frac{1}{2}$ is the only solution of (QC) with $\theta < m(q)$. Here $\tilde{v}(x) = -x|x|$. For $(q, \theta) \in \Gamma$ with $\theta < m$ and $u < q$, $F^{-1}(u) \leq \theta < m$, so the integrand in (QC) is

$$-(m - F^{-1}(u))^2 + (\theta - F^{-1}(u))^2 + 2(m - \theta)(m - F^{-1}(u)) = (m - \theta)^2,$$

and the left-hand side of (QC) equals $(m-\theta)^2(2q-1)$. So $q = \frac{1}{2}$ is the only solution with $\theta < m(q)$. If $\overline{q} > \frac{1}{2}$, then $(\frac{1}{2}, F^{-1}(\frac{1}{2})) \in \Gamma$ with $F^{-1}(\frac{1}{2}) < m(\frac{1}{2})$ solves it, and $(G_{1/2}, G_{1/2})$ is an equilibrium by Lemma 10 (ii). If $\overline{q} \leq \frac{1}{2}$, then $G_{1/2} = F$, and $w'_F(x) = -2\int |x - t| dF(t)$ has right derivative $2(1 - 2F(x)) \geq 0$ on $[\underline{\theta}, \overline{\theta})$, as $F(x) \leq F(\overline{\theta}-) = \overline{q} \leq \frac{1}{2}$ there. So (F, F) is an equilibrium by Step 2 of the proof of Lemma 10. If $\underline{\theta} = \overline{\theta}$, the claim is trivial.

Uniqueness. Let $\theta_0 := F^{-1}(\frac{1}{2})$, $m := m(\frac{1}{2})$, $\ell := m - \theta_0$, $b := \frac{1}{2} - F(\theta_0-)$ and $w := w_{G_{1/2}}$. As w' has right derivative $2\big(1 - 2G_{1/2}(x)\big)$ and left derivative $2\big(1 - 2G_{1/2}(x-)\big)$, and $G_{1/2}$ is below $\frac{1}{2}$ on $[0, \theta_0)$, equal to $\frac{1}{2}$ on $[\theta_0, m)$ and to 1 from m on, w is strictly convex on $[0, \theta_0]$, affine on $[\theta_0, m]$ and strictly concave on $[m, 1]$. Let π be as in Step 1 of the proof of Lemma 10, for $K = G_{1/2}$ and $\theta = \theta_0$: as w is affine on $[\theta_0, m]$, π equals w on $[0, m]$ and its tangent at m on $[m, 1]$. A best reply G' to $G_{1/2}$ puts no mass on $(m, 1]$, where $\pi > w$, and equals F on $[0, \theta_0)$, where $d\pi' > 0$; its remaining mass $\frac{1}{2} + b$ lies on $[\theta_0, m]$, with $\int_{[\theta_0, m]} (t - \theta_0)\, dG'(t) = \frac{\ell}{2}$. If $b = 0$ or $\ell = 0$, then $G' = G_{1/2}$. Otherwise, let L be the

tangent to $w_{G'}$ at m. As $w'_{G'}(z) = -2\int |z-t|\, dG'(t)$ and G' puts no mass above m,

$$L(\theta_0) - w_{G'}(\theta_0) = \int_{\theta_0}^{m} \big(w'_{G'}(z) - w'_{G'}(m)\big)dz = \ell^2 - 2\int_{[\theta_0,m]} (t-\theta_0)^2 dG'(t) \geq 0,$$

as $(t-\theta_0)^2 \leq \ell(t-\theta_0)$ on $[\theta_0, m]$, with equality only if G' charges only θ_0 and m there, which by the two moment conditions means $G' = G_{1/2}$. If $G' \neq G_{1/2}$, then $G_{1/2-\varepsilon}$, which moves a mass $\varepsilon \leq b$ from θ_0 into the pool at $m(\frac{1}{2}-\varepsilon)$, gains $\varepsilon\big(L(\theta_0) - w_{G'}(\theta_0)\big) + O(\varepsilon^2) > 0$ over $G_{1/2}$ against G'. As equilibria form a product set containing $(G_{1/2}, G_{1/2})$, every equilibrium signal is a best reply to $G_{1/2}$ and has $G_{1/2}$ as a best reply, so it is $G_{1/2}$.

C.5 Full revelation and the top atom

Lemma 11. *Let $v(x)$ be well S-shaped and F a prior on $[0,1]$ with $\underline{\theta} < \overline{\theta}$. If F is a best reply to F, then F has an atom at $\overline{\theta}$ of mass $\alpha := F(\overline{\theta}) - F(\overline{\theta}-) \geq \frac{1}{2}$.*

Proof. Let $w := w_F$, $g := w'$ and $L(y) := w(\overline{\theta}) - w'(\overline{\theta})(\overline{\theta} - y)$, the tangent to w at $\overline{\theta}$, with w_F as in the proof of Lemma 10. Throughout, I use that for $0 \leq b \leq a \leq 1$,

$$R(a,b) := \tilde{v}(a) - \tilde{v}(b) - \tilde{v}'(a)(a-b) \geq \tilde{v}(a-b) - (a-b)\tilde{v}'(a-b) \geq -\tilde{v}(a-b). \qquad \text{(R)}$$

Indeed, let $\delta := a - b$. Then $R(a,b) = \int_0^\delta [\tilde{v}'(b+t) - \tilde{v}'(a)]\, dt$. As $\tilde{v}'$ is concave on $[0,1]$, $\tilde{v}'(t+b) - \tilde{v}'(t)$ is non-increasing in t, so for $t \in [0,\delta]$, $\tilde{v}'(b+t) - \tilde{v}'(a) \geq \tilde{v}'(t) - \tilde{v}'(\delta)$, and hence $R(a,b) \geq \int_0^\delta [\tilde{v}'(t) - \tilde{v}'(\delta)]\, dt = \tilde{v}(\delta) - \delta\tilde{v}'(\delta)$, which is at least $-\tilde{v}(\delta)$ as $\mathcal{E}_{\tilde{v}}(\delta) \geq 2$ (Lemma 3).

Case 1: $\overline{\theta}$ is isolated in $\operatorname{supp}(F)$. Let $t := \max\big(\operatorname{supp}(F) \cap [0, \overline{\theta})\big)$. Then $\alpha > 0$, $\delta := \overline{\theta} - t \in (0,1]$ and $F((t,\overline{\theta})) = 0$. First, $w \geq L$ on $\operatorname{supp}(F) \cap [0,\overline{\theta})$. Let $B \subseteq [0,\overline{\theta})$ be a Borel set with $\beta := F(B) > 0$ and $b := \mathbb{E}_F[\theta | \theta \in B]$. For $\varepsilon \in (0,\beta]$, the signal that pools the atom at $\overline{\theta}$ with a share $\frac{\varepsilon}{\beta}$ of the types in B at $m_\varepsilon := \frac{\alpha\overline{\theta} + \varepsilon b}{\alpha + \varepsilon}$ and reveals all other types

is a mean-preserving contraction of F. Its gain over F is

$$(\alpha+\varepsilon)w(m_\varepsilon)-\alpha w(\overline{\theta})-\frac{\varepsilon}{\beta}\int_B w\,dF=\frac{\varepsilon}{\beta}\int_B(L-w)\,dF+o(\varepsilon),$$

since $m_\varepsilon-\overline{\theta}=-\frac{\varepsilon(\overline{\theta}-b)}{\alpha+\varepsilon}$, w is differentiable at $\overline{\theta}$, and $L(b)=\frac{1}{\beta}\int_B L\,dF$ as L is affine. So $\int_B(w-L)\,dF\geq 0$ for every such B, i.e. $w\geq L$ F-a.e. on $[0,\overline{\theta})$, and on $\mathrm{supp}(F)\cap[0,\overline{\theta})$ by continuity. In particular, as $\tilde{v}$ is odd and $\tilde{v}(0)=\tilde{v}'(0)=0$,

$$0\leq w(t)-L(t)=-\alpha\tilde{v}(\delta)-\int_{[0,t]}R(\overline{\theta}-y,t-y)\,dF(y)\leq-\alpha\tilde{v}(\delta)-(1-\alpha)\big(\tilde{v}(\delta)-\delta\tilde{v}'(\delta)\big),$$

by (R) with $a=\overline{\theta}-y$, $b=t-y$. Dividing by $-\tilde{v}(\delta)>0$ gives $\alpha\geq(1-\alpha)(\mathcal{E}_{\tilde{v}}(\delta)-1)$, i.e. $\alpha\geq 1-\frac{1}{\mathcal{E}_{\tilde{v}}(\delta)}$, which is at least $\frac{1}{2}$ by Lemma 3.

Case 2: $\overline{\theta}$ is not isolated. Then $\overline{\theta}$ is a limit point of $\mathrm{supp}(F)\cap[0,\overline{\theta})$, so $F([0,\overline{\theta}))=1-\alpha>0$. Suppose that $g'_-(\overline{\theta})<0$. As the left derivative of the concave g is left-continuous, g is strictly decreasing on some $[\overline{\theta}-\eta,\overline{\theta}]$, so w is strictly concave there. This interval contains infinitely many points of $\mathrm{supp}(F)$, so pooling all types in it at their mean is feasible and, by the strict Jensen inequality, strictly better than F. Hence F is a best reply only if $g'_-(\overline{\theta})\geq 0$. For $h\in(0,1)$, $\frac{g(\overline{\theta})-g(\overline{\theta}-h)}{h}=\int\frac{\tilde{v}'(\overline{\theta}-y)-\tilde{v}'(\overline{\theta}-y-h)}{h}\,dF(y)$, and as $\tilde{v}'$ is concave, each integrand is non-increasing as $h\downarrow 0$ and converges to $\tilde{v}''_-(\overline{\theta}-y)$, so

$$g'_-(\overline{\theta})=\alpha|\tilde{v}''_+(0)|-\int_{[0,\overline{\theta})}|\tilde{v}''_-(\overline{\theta}-y)|\,dF(y),$$

by monotone convergence, using $\tilde{v}''_-(0)=|\tilde{v}''_+(0)|$ and $\tilde{v}''_-(z)<0$ for $z\in(0,1]$ (property 2) in Appendix A.1). So F is a best reply only if $\alpha|\tilde{v}''_+(0)|\geq\int_{[0,\overline{\theta})}|\tilde{v}''_-(\overline{\theta}-y)|\,dF(y)$. As $-\tilde{v}'$ is convex on $[0,1]$ with $-\tilde{v}'(0)=0$, $|\tilde{v}''_-(z)|\geq|\tilde{v}''_+(0)|$ and $|\tilde{v}''_-(z)|>0$ for $z\in(0,1]$, so this requires $\tilde{v}''_+(0)\neq 0$ and $\alpha\geq 1-\alpha$. □

C.6 Proof of the bounds in Proposition 9

Write $x_u := F^{-1}(u)$, $p := \mathbb{E}_F[\theta]$ and $\overline{q} := F(\overline{\theta}-)$, so that $G_q = F$ for every $q \geq \overline{q}$ (Appendix C.2). If $\underline{\theta} = \overline{\theta}$, every feasible signal is $\{\overline{\theta}\} = G_0$, and the firm's utility is $\overline{\theta}$ under every rule; so let $\underline{\theta} < \overline{\theta}$, hence $\overline{q} > 0$. Let (G, G) be a symmetric equilibrium with $G = G_{q_0}$, and let $q := \min(q_0, \overline{q})$, so that $G = G_q$, and $q < \overline{q}$ unless $G = F$.

Step 1: $q \leq \frac{1}{2}$. If $G = F$, then $q = \overline{q} = 1 - \alpha \leq \frac{1}{2}$ by Lemma 11. Let $q < \overline{q}$, so $x_{q^+} < m := m(q) < \overline{\theta}$, and let $w := w_{G_q}$. I show that G_q solves (QC). As G_q is a best reply to G_q and w is Lipschitz, as v' is bounded, there is a convex $\phi \geq w$ on $[\underline{\theta}, \overline{\theta}]$ with $\int \phi \, dF = \int w \, dG_q$ (Dworczak and Martini, 2019, Theorem 2). Then $\int w \, dG_q \leq \int \phi \, dG_q \leq \int \phi \, dF$ are equalities, so $\phi = w$ G_q-a.e., and Jensen's inequality $(1-q)\phi(m) \leq \int_q^1 \phi(x_u)\, du$ is an equality. As the law of x_U given $U > q$ has x_{q^+} and $\overline{\theta}$ as the infimum and supremum of its support, ϕ coincides on $(x_{q^+}, \overline{\theta})$ with an affine L. Since $\phi \geq w$ with equality at the atom m of G_q, L is the tangent to w at m, and $\phi \geq L$ on $[\underline{\theta}, \overline{\theta}]$, as ϕ is convex. Let $\Delta(\theta) := L(\theta) - w(\theta)$, the left-hand side of (QC) at (q, θ). On $(x_{q^+}, m]$, $\Delta = \phi - w \geq 0$, so $\Delta(x_{q^+}) \geq 0$. Moreover, $\phi(x_q) = w(x_q)$: if $x_q > \underline{\theta}$, then ϕ is continuous at $x_q < \overline{\theta}$, $\phi(x_u) = w(x_u)$ for almost every $u < q$, and $x_u \to x_q$ as $u \uparrow q$; if $x_q = \underline{\theta}$, then G_q has an atom of mass at least q at $\underline{\theta}$. If $q = 0$, then $x_{0^+} = \underline{\theta}$, $m = p$ and $w(\theta) = \tilde{v}(\theta - p)$, so $L \equiv 0$ and $\Delta(\underline{\theta}) = \tilde{v}(p - \underline{\theta}) < 0$, contradicting $\Delta(\underline{\theta}) \geq 0$; so $q > 0$. So $\Delta(x_q) \leq \phi(x_q) - w(x_q) = 0$, and $\Delta(\theta) = 0$ for some $\theta \in [x_q, x_{q^+}]$, where $\theta < m$.

So (q, θ) solves (QC). Let $d := m - \theta \in (0, 1]$. By (QC), $E(q) := \int_0^q R(m - x_u, \theta - x_u)\, du + (1-q)\tilde{v}(d) = 0$, with R as in (R), and $0 \leq \theta - x_u \leq m - x_u \leq 1$ for $u < q$. By (R), $R(m - x_u, \theta - x_u) \geq -\tilde{v}(d)$, and since $\tilde{v}(d) < 0$, this gives $E(q) \geq -(2q-1)\tilde{v}(d) > 0$ for $q > \frac{1}{2}$, so every root satisfies $q \leq \frac{1}{2}$.

Step 2: accuracy. For any gap rule and any symmetric profile, as $v(x) - \frac{1}{2}$ is odd, $s_1 v(s_1 - s_2) + s_2 v(s_2 - s_1) = \frac{s_1+s_2}{2} + h_v(|s_1 - s_2|)$ with $h_v(\delta) := \delta\big(v(\delta) - \frac{1}{2}\big)$, so $u_F = p + \mathbb{E}[h_v(|s_1 - s_2|)]$. By Lemma 6, $h_v(\delta) \leq h^*(\delta) := \delta^2 - \frac{\delta^3}{2}$ on $[0, 1]$, with equality

at some $\delta \in (0,1)$ only if $v = v^*$, and at $\delta = 1$ if and only if $v(1) = 1$. Hence $u_F \le U(q) := p + \mathbb{E}_{G_q \times G_q}[h^*(|s_1 - s_2|)]$, with equality if and only if $h_v(|s_1 - s_2|) = h^*(|s_1 - s_2|)$ almost surely.

Step 3: U is non-decreasing on $[0, \frac{1}{2}]$, and $U(q) < U(\frac{1}{2})$ for $q < \min(\frac{1}{2}, \overline{q})$. In this step, $\theta := x_q$, $m := m(q)$ and $d := m - \theta$, so $d > 0$ if and only if $q < \overline{q}$. $U(q) - p = \int_0^q \int_0^q h^*(|x_u - x_w|)\, du\, dw + 2(1-q)\int_0^q h^*(m - x_u)\, du$. As $(1-q)m(q) = \int_q^1 x_u\, du$, m and hence U are Lipschitz on $[0, \frac{1}{2}]$, and at every continuity point q of x_u, hence for almost every q, $m'(q) = \frac{d}{1-q}$, and $\frac{1}{2}U'(q) = -\int_0^q R^*(m - x_u, \theta - x_u)\, du + (1-q)h^*(d)$, $R^*(a,b) := h^*(a) - h^*(b) - h^{*\prime}(a)(a-b)$. As h^* is a cubic, $R^*(a, a-d) = -d^2 + \frac{3}{2}ad^2 - \frac{d^3}{2}$, so, with $\mu := \frac{1}{q}\int_0^q x_u\, du$, $\frac{1}{2}U'(q) = d^2[1 + \frac{d}{2}(2q-1) - \frac{3}{2}q(m-\mu)] \ge d^2[1 - \frac{1-2q}{2} - \frac{3}{2}q] = d^2\,\frac{1-q}{2}$, using $d \le 1$, $m - \mu \le 1$ and $q \le \frac{1}{2}$. So $U' \ge 0$ a.e. on $[0, \frac{1}{2}]$, and $U' > 0$ a.e. on $(0, \min(\frac{1}{2}, \overline{q}))$, which proves Step 3. For $q \in [\overline{q}, \frac{1}{2}]$, $G_q = G_{1/2}$ and $U(q) = U(\frac{1}{2})$.

Conclusion. Under $v^*(x)$, $h_{v^*} = h^*$, and $(G_{1/2}, G_{1/2})$ is an equilibrium (Appendix C.4). Hence $U(\frac{1}{2})$ is the firm's utility in the median-censorship equilibrium of $v^*(x)$. By Steps 1–3, $u_F \le U(q) \le U(\frac{1}{2})$.

C.7 A convexity lemma

The following lemma extends the converse of the Hermite–Hadamard inequality (see Niculescu and Persson (2003)), which is the case of a uniform K.

Lemma 12. *Let a CDF K and a function h be continuous on an interval J, with K strictly increasing. If $h\big(\mathbb{E}_K[\theta|\theta \in J']\big) \le \mathbb{E}_K[h(\theta)|\theta \in J']$ for every interval $J' \subseteq J$, then h is convex.*

Proof. Let $x < z$ in J, let ℓ be the secant of h through x and z, and suppose that $g := h - \ell$ has a positive maximum on $[x,z]$, first attained at $c \in (x,z)$. For $\delta > 0$ small enough that $\mathbb{E}_K[\theta|\theta \in [c-\delta, z]] > c$, there is $\delta' \in (0, z-c)$ such that $J' := [c-\delta, c+\delta']$ satisfies $\mathbb{E}_K[\theta|\theta \in J'] = c$, as K is continuous and strictly increasing. Since $g < g(c)$ on $[c-\delta, c)$,

which has positive K-mass, and ℓ is affine, $\mathbb{E}_K[h(\theta)|\theta \in J'] = \ell(c) + \mathbb{E}_K[g(\theta)|\theta \in J'] < h(c)$, a contradiction. Hence $h \leq \ell$ on $[x, z]$. □

Lemma 13. *Let F be a prior, $\underline{\theta} = \min \operatorname{supp}(F)$, $\overline{\theta} = \max \operatorname{supp}(F)$, and let V be bounded and upper semicontinuous on $[\underline{\theta}, \overline{\theta}]$. If $\int V\, dG \leq \int V\, dF$ for every $G \leq_{cx} F$, then there is a bounded convex $\phi \geq V$ on $[\underline{\theta}, \overline{\theta}]$ with $\phi = V$ on $\operatorname{supp}(F)$. In particular, if $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$, then V is convex on $[\underline{\theta}, \overline{\theta}]$.*

Proof. Let $S := \operatorname{supp}(F)$, let $x < y < z$ with $x, z \in S$, and let $y = \lambda x + (1-\lambda)z$. For $0 < \varepsilon < \min(y-x, z-y)$, let $A := [x-\varepsilon, x+\varepsilon]$ and $B := [z-\varepsilon, z+\varepsilon]$; then $F(A), F(B) > 0$, and the conditional means $m_A \in A$, $m_B \in B$ satisfy $m_A < y < m_B$. Choose $\alpha, \beta \in (0, 1]$ with $\alpha \int_A \theta\, dF + \beta \int_B \theta\, dF = (\alpha F(A) + \beta F(B))\, y$, and let G be obtained from F by moving the mass $\alpha F|_A + \beta F|_B$ to an atom at y. By Jensen's inequality $G \leq_{cx} F$, so with $\lambda_\varepsilon := \alpha F(A)/(\alpha F(A) + \beta F(B))$, $V(y) \leq \lambda_\varepsilon \sup_A V + (1 - \lambda_\varepsilon) \sup_B V$. As $\varepsilon \to 0$, $\lambda_\varepsilon \to \lambda$, and by upper semicontinuity $V(y) \leq \lambda V(x) + (1-\lambda)V(z)$. Let $\tilde{V} := V$ on S and $\tilde{V} := +\infty$ elsewhere, and let $\phi(y) := \inf \sum_i \lambda_i \tilde{V}(x_i)$ over finite convex combinations with $\sum_i \lambda_i x_i = y$, the convex hull of $\tilde{V}$. For fixed points x_i, the minimum over the polytope $\{\lambda \geq 0 : \sum_i \lambda_i = 1,\ \sum_i \lambda_i x_i = y\}$ is attained at a vertex, which has at most two positive entries. So ϕ is convex, $\phi \geq V$ on $[\underline{\theta}, \overline{\theta}]$ by the chord inequality, $\phi \leq V$ on S, and $\inf V \leq \phi \leq \sup V$, as $\underline{\theta}, \overline{\theta} \in S$. If $S = [\underline{\theta}, \overline{\theta}]$, then $V = \phi$. □

C.8 Proofs of the benchmark results

Throughout, a rule is a general selection rule, and $V_G(x) = \int H(x, y)\, dG(y)$ is the agent's value function against a rival who plays G. For a CDF K, let $I_K(x) := \int_{-\infty}^{x} K(t)\, dt$; for distributions with the same mean, $G \leq_{cx} F$ if and only if $I_G \leq I_F$.

C.8.1 Two tools

Lemma 14. *In every symmetric profile (G, G), the firm's utility is $u_F = p + 2\operatorname{Cov}_G\big(s, V_G(s)\big)$.*

Proof. With s_1, s_2 i.i.d. G, $u_F = \mathbb{E}[s_2] + \mathbb{E}[(s_1 - s_2)H(s_1, s_2)]$. Since $\kappa := H - \frac{1}{2}$ is anti-symmetric, $\mathbb{E}[s_2\kappa(s_1, s_2)] = -\mathbb{E}[s_1\kappa(s_1, s_2)]$, so $\mathbb{E}[(s_1 - s_2)H(s_1, s_2)] = 2\mathbb{E}[s_1\kappa(s_1, s_2)] = 2\mathbb{E}\left[s_1\left(V_G(s_1) - \frac{1}{2}\right)\right]$, and $\mathbb{E}[V_G(s)] = \frac{1}{2}$. □

Lemma 15. *Let F be any prior and $V : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. There is a rule with $V_F = V$ on* $\mathrm{supp}(F)$ *if and only if $V(\theta) \leq_{cx} U[0, 1]$ for $\theta \sim F$, i.e.,*

$$\mathbb{E}_F[V(\theta)] = \tfrac{1}{2} \qquad \textit{and} \qquad \mathbb{E}_F\left[(V(\theta) - t)^+\right] \leq \tfrac{(1-t)^2}{2} \ \textit{for all } t \in [0, 1].$$

Then one such rule is $H(x, y) = \mathbb{P}(S_x > S_y) + \frac{1}{2}\mathbb{P}(S_x = S_y)$, where the agents' scores are independent, the score S_x of posterior mean x has mean $V(x)$, and $S_\theta \sim U[0, 1]$ for $\theta \sim F$.

Proof. Necessity: $\mathbb{E}_F[V] = \frac{1}{2}$ by symmetry. For every set A, the winner's type lies in A with probability $2\int_A V\, dF$, at most the probability $1 - (1 - F(A))^2$ that some type lies in A. Applied to the upper level sets of V, this bounds the sum of the top m quantiles of $V(\theta)$ by $m - \frac{m^2}{2}$, the same sum for $U[0, 1]$, at every mass m of such a set, and hence at every $m \in [0, 1]$: between two such masses, $V(\theta)$ has an atom, so the left side is linear in m, while the right side is concave. With equal means, this is the stated condition. Sufficiency: by Strassen's theorem, there is a kernel $\kappa(\cdot|z)$ with mean z that spreads the distribution of $V(\theta)$ into $U[0, 1]$; let $S_x \sim \kappa(\cdot|V(x))$, and for x with $V(x)$ outside the support of the law of $V(\theta)$, let S_x be any law on $[0, 1]$ with mean $V(x)$. Against a fully revealing rival, whose score is $U[0, 1]$, $\mathbb{P}(S_x > S_y) = \mathbb{E}[S_x] = V(x)$, and the rule is symmetric and constant-sum by construction. □

The second lemma is the feasibility claim in Subsection 5.2: noise only makes winning probabilities less dispersed than exact ranking, and the noiseless rule is the boundary case, with $V_F = F$ and $F(\theta) \sim U[0, 1]$.

C.8.2 Proof of Proposition 8

I first show that, for priors (p, p), H attains the first best $2p - p^2$ in some equilibrium if and only if $H(1,0) = 1$ and $(1-p)H(x,0) + p\,H(x,1) \leq \frac{1-p+x}{2}$ for all $x \in (0,1)$. For any signals, $u_F = \mathbb{E}[s_2 + (s_1 - s_2)H(s_1, s_2)] \leq \mathbb{E}[\max(s_1, s_2)] \leq \mathbb{E}[s_1 + s_2 - s_1 s_2] = 2p - p^2$, where the second inequality holds as $\mathbb{E}[\max(\theta_1, \theta_2)|s_1, s_2] = s_1 + s_2 - s_1 s_2$, and it is an equality if and only if $\min(s_1, s_2)\big(1 - \max(s_1, s_2)\big) = 0$ almost surely. If both agents put mass on $(0,1)$, this fails with positive probability, so in an equilibrium that attains the first best, some agent, say agent 1, reveals fully. As the game is symmetric and constant-sum, its equilibrium strategies form the same set for both agents, so $(\{0,1\};\{0,1\})$ is an equilibrium too. The first inequality is an equality only if $H(1,s) = 1$ for almost every posterior $s < 1$ of agent 2, and agent 2 sends some $s < 1$, as $p < 1$. As H is nondecreasing in the own posterior and constant-sum, $H(1,0) = 1 - H(0,1) \geq 1 - H(s,1) = H(1,s) = 1$. Against a fully revealing rival, the agent's value function is $V(x) = (1-p)H(x,0) + p\,H(x,1)$, with $V(0) = \frac{1-p}{2}$ and $V(1) = 1 - \frac{p}{2}$ when $H(1,0) = 1$, so full revelation is a best reply if and only if V lies below its chord from 0 to 1, which is $\frac{1-p+x}{2}$. Conversely, if $H(1,0) = 1$ and this holds, $(\{0,1\};\{0,1\})$ is an equilibrium, and the firm selects a good agent over a bad one for sure, which attains $2p - p^2$.

(i) This condition is linear in p, so it holds for all $p \in (0,1)$ if and only if it holds in the limits $p \to 0$ and $p \to 1$, which give $H(x,0) \leq \frac{1+x}{2}$ and $H(x,1) \leq \frac{x}{2}$. If both are strict on $(0,1)$, against a fully revealing rival with any prior p_j, $V(x) < \frac{1-p_j+x}{2}$ on $(0,1)$ while V equals the chord at 0 and 1, so full revelation is the unique best reply for every own prior, as every other signal puts mass on $(0,1)$. So $(\{0,1\};\{0,1\})$ is an equilibrium for all $(p_1, p_2) \in (0,1)^2$; if (G_1, G_2) is another, by interchangeability so is $(G_1, \{0,1\})$, so G_1 is a best reply to full revelation, and $G_1 = \{0,1\}$, and likewise $G_2 = \{0,1\}$.

(ii) The pass/fail rule is nondecreasing in the own posterior, $H(1,0) = 1$, $H(x,0) =$

$\frac{1}{2} < \frac{1+x}{2}$ and $H(x,1) = 0 < \frac{x}{2}$ for $x \in (0,1)$, so (i) applies.

C.8.3 Uniqueness under the noiseless rule

Let $\text{supp}(F) = I := [\underline{\theta}, \overline{\theta}]$, and let F be convex on $[\underline{\theta}, \overline{\theta})$ with no atom except possibly one of mass $m \geq 0$ at $\overline{\theta}$. For a distribution K, let $\bar{K}(x) := K(x^-) + \frac{1}{2}K(\{x\})$ be its mid-CDF, the payoff of a posterior mean x against a rival who plays K under the noiseless rule. A signal G is feasible if and only if $I_G \leq I_F$ on I with equality at $\overline{\theta}$; then $\int_x^{\overline{\theta}} G \geq \int_x^{\overline{\theta}} F$ for $x < \overline{\theta}$, so $G(\overline{\theta}-) \geq F(\overline{\theta}-)$ and $G(\{\overline{\theta}\}) \leq m$. Let $\phi = \phi_c + j\mathbb{1}\{\overline{\theta}\}$, with ϕ_c convex and continuous on I and a jump $j \geq 0$ at $\overline{\theta}$. Integrating by parts twice,

$$\int \phi \, dF - \int \phi \, dG = \int \big(I_F(y) - I_G(y)\big) d\phi_c'(y) + j\big(m - G(\{\overline{\theta}\})\big),$$

and both terms are nonnegative.

Let $(F, \hat{G})$ be an equilibrium. As all equilibria form a symmetric product set, $(\hat{G}, \hat{G})$ and $(\hat{G}, F)$ are equilibria too, and each agent gets $\frac{1}{2}$ in all of them, so $\int \bar{\hat{G}} \, d\hat{G} = \int \bar{\hat{G}} \, dF = \int \bar{F} \, d\hat{G} = \int \bar{F} \, dF = \frac{1}{2}$.

First, $\hat{G}$ has no atom in $[\underline{\theta}, \overline{\theta})$. An atom a at $\underline{\theta}$ would give $I_{\hat{G}}(x) \geq a(x - \underline{\theta}) > I_F(x)$ for x close to $\underline{\theta}$, as F is continuous at $\underline{\theta}$ with $F(\underline{\theta}) = 0$. If $\hat{G}$ had an atom $a > 0$ at $x_0 \in (\underline{\theta}, \overline{\theta})$, then $I_{\hat{G}}(x_0) < I_F(x_0)$, since otherwise $I_F - I_{\hat{G}} \geq 0$ would have a minimum at x_0 with left derivative $F(x_0) - \hat{G}(x_0^-)$ above its right derivative $F(x_0) - \hat{G}(x_0)$. So replacing the atom by masses $\frac{2a}{3}$ at $x_0 + \varepsilon$ and $\frac{a}{3}$ at $x_0 - 2\varepsilon$ is feasible for small ε, as $I_F - I_{\hat{G}} > 0$ near x_0, and against $\hat{G}$ it raises the payoff by $a^2(\frac{2}{3} - \frac{1}{2}) + o(1) > 0$, contradicting that $\hat{G}$ is a best reply to $\hat{G}$. So $\bar{\hat{G}} = \hat{G}$ on $[\underline{\theta}, \overline{\theta})$, and $\bar{\hat{G}}$ jumps at $\overline{\theta}$ by $\frac{1}{2}\hat{G}(\{\overline{\theta}\})$, which is zero if $m = 0$.

Second, $\hat{G}$ is convex on $[\underline{\theta}, \overline{\theta})$. As F is a best reply to $\hat{G}$, pooling the types in an interval $J \subseteq [\underline{\theta}, \overline{\theta})$ into their mean m_J does not pay, i.e., $\hat{G}(m_J) \leq \mathbb{E}_F[\hat{G}(\theta) | \theta \in J]$; as F is convex with support I, it is continuous and strictly increasing on $[\underline{\theta}, \overline{\theta})$, and Lemma 12 applies with $K = F$ and $h = \hat{G}$ on every $[\underline{\theta}, x]$ with $x < \overline{\theta}$.

Hence $\bar{F}$ and $\bar{\hat{G}}$ both have the form of ϕ, with jumps $\frac{m}{2}$ and $\frac{1}{2}\hat{G}(\{\overline{\theta}\})$ at $\overline{\theta}$. The equalities above give $\int \bar{F}\, d(F-\hat{G}) = 0$ and $\int \bar{\hat{G}}\, d(F-\hat{G}) = 0$, so all four nonnegative terms of the identity vanish. If $m > 0$, the jump term for $\bar{F}$ gives $\hat{G}(\{\overline{\theta}\}) = m$. The other terms give $I_F = I_{\hat{G}}$ on the supports in $(\underline{\theta}, \overline{\theta})$ of dF' and $d\hat{G}'$, and there $F = \hat{G}$, since each such point is a minimum of $I_F - I_{\hat{G}} \geq 0$, which is differentiable on $(\underline{\theta}, \overline{\theta})$, as F and $\hat{G}$ are continuous there. On each open interval of $(\underline{\theta}, \overline{\theta})$ off these supports, F and $\hat{G}$ are affine, and they agree at its endpoints: at points of the supports, at $\underline{\theta}$, where $F = \hat{G} = 0$, and at $\overline{\theta}$, where $F(\overline{\theta}-) = 1 - m = \hat{G}(\overline{\theta}-)$. Hence $\hat{G} = F$, and (F, F) is the unique equilibrium.

C.8.4 The first best for a general support

Lemma 16. *Let F be any prior, $S := \operatorname{supp}(F) \subseteq [\underline{\theta}, \overline{\theta}]$, and $\bar{F}(x) := F(x^-) + \frac{1}{2}F(\{x\})$.*

(i) If $\bar{F}$ coincides on S with some function convex on $[\underline{\theta}, \overline{\theta}]$, the rule that rounds each posterior mean down to the support, $r(x) := \max\big(S \cap [\underline{\theta}, x]\big)$, and selects the higher rounded value, tossing a coin at ties, attains the first best, with a fully revealing equilibrium.

(ii) If a general selection rule nondecreasing in the own posterior mean attains the first best in a fully revealing equilibrium, $\bar{F}$ coincides on S with a function convex on $[\underline{\theta}, \overline{\theta}]$.

(iii) If $S = [\underline{\theta}, \overline{\theta}]$, every equilibrium of every general selection rule that attains the first best is fully revealing, and some general selection rule nondecreasing in the own posterior mean attains the first best in some equilibrium if and only if F is convex on $[\underline{\theta}, \overline{\theta})$ and has no atom below $\overline{\theta}$; the noiseless rule then attains it, with a fully revealing equilibrium.

Proposition 10 is part (iii), which also shows that every equilibrium that attains the first best is fully revealing, with the uniqueness claim proved in Appendix C.8.3, and parts (i) and (ii) extend it to supports that are not intervals. Every binary prior

satisfies the condition of part (i), and the rounding rule is then the pass/fail rule of Proposition 8.

Proof. Here $\bar{F}(x)$ is the winning probability of type x against a fully revealing rival when the higher type is selected and ties are split, as symmetry forces. In terms of F, the condition in (i) and (ii) says that F is convex on every interval contained in S, every atom of F below $\overline{\theta}$ is an isolated point of S, and that the chords of $\bar{F}$ across the gaps of S have nondecreasing slopes.

(i) *Sufficiency, for any support.* Let ψ be convex with $\psi = \bar{F}$ on S, and let ϕ equal $\bar{F}$ on S and be affine on each gap (a, b) of S. Then ϕ is convex: on each interval contained in S it equals ψ, and at the ends of a gap its slopes are ordered, as the chord of the convex ψ over $[a, b]$ has slope between $\psi'(a^+)$ and $\psi'(b^-)$, and consecutive chords of ψ have nondecreasing slopes. As $\bar{F}$ is nondecreasing on S, $\phi \geq \bar{F}(a)$ on a gap (a, b). Let $r(x) := \max\left(S \cap [\underline{\theta}, x]\right)$ and let the rule select the higher rounded posterior mean, tossing a coin at ties; it is nondecreasing in the own posterior mean. Against a fully revealing rival, $V_F = \bar{F}$ on S and $V_F(x) = \bar{F}(r(x)) \leq \phi(x)$ off S. So for every $G \leq_{cx} F$, $\int V_F \, dG \leq \int \phi \, dG \leq \int \phi \, dF = \int V_F \, dF$, and full revelation is an equilibrium. Distinct types are never rounded to the same point, so the higher type is always selected.

(ii) *Necessity, for any support and a rule nondecreasing in the own posterior mean.* Under full revelation, the first best requires $H(x, y) = 1$ for $F \otimes F$-almost all $x > y$, and $H(x, x) = \frac{1}{2}$ by symmetry, so $V_F = \bar{F}$ F-almost everywhere and at every atom; as V_F is nondecreasing, $V_F(x) \in [F(x^-), F(x)]$ for every $x \in S$. Three deviations, each a mean-preserving contraction of F, rule out the three ways in which $\bar{F}|_S$ can fail to extend convexly. (a) An atom $a < \overline{\theta}$ of mass m that is a limit of S from the right: pooling a fraction α of the atom with all types in $(a, a + \delta)$, of mass $\mu_\delta > 0$, gives a posterior mean in $(a, a + \delta)$ with winning probability at least $F(a)$, so the atom gains at least $\alpha \frac{m}{2}$, while the other types lose at most $\mu_\delta(F(a + \delta) - F(a))$, which is smaller for

small δ and $\alpha = \mu_\delta$. (b) An atom $a < \overline{\theta}$ that is a limit of S from the left only: then a is the left end of a gap (a, b), and pooling the types in $(a - \delta, a)$, of mass μ_δ, with a mass $w = O(\mu_\delta \delta)$ of the types at b into the posterior mean a raises the former from at most $F(a^-)$ to $\bar{F}(a) = F(a^-) + \frac{m}{2}$ and costs the latter at most w. (c) Points $x < y < z$ of S with $\bar{F}(y)$ above the chord of $\bar{F}$ through x and z: pooling types near x and near z (at x and z themselves if they are atoms) into the posterior mean y in the right proportions pays, as V_F is continuous at non-atoms of S by (a) and (b). If none of (a)–(c) applies, $\bar{F}|_S$ is convex on S and continuous at every limit point of S below $\overline{\theta}$, and its extension by chords is convex. This completes the proof of (ii).

(iii) *Interval support.* Let $S = [\underline{\theta}, \overline{\theta}]$ and let (G_1, G_2) attain the first best. The selected type must be the higher type almost surely, and selection depends only on the posterior means, so for almost all pairs of posterior means the conditional supports of θ_1 and θ_2 are ordered. If a realization of agent 1 pooled two types $a < b$ of S, the conditional supports of agent 2 would avoid (a, b), whose F-mass is positive. So both agents reveal fully, and (ii) applies: by (a), F has no atom below $\overline{\theta}$, and by (c), F is convex on $[\underline{\theta}, \overline{\theta})$. Conversely, then $\bar{F} = F$ on $[\underline{\theta}, \overline{\theta})$, and an atom at $\overline{\theta}$ only raises $\bar{F}(\overline{\theta})$ above $F(\overline{\theta}-)$, which keeps $\bar{F}$ convex on $[\underline{\theta}, \overline{\theta}]$; with no gaps, the rounding rule of (i) is the noiseless rule. □

Gap rules. With $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$, a gap rule attains the first best only if it is the noiseless rule. Let $d = s_1 - s_2$. For any gap rule and feasible G_1, G_2, since $v(-x) = 1 - v(x)$,

$$u_F = \mathbb{E}[\max(s_1, s_2)] - \mathbb{E}\big[|d|\big(1 - v(|d|)\big)\big] \le \mathbb{E}[\max(s_1, s_2)] \le \mathbb{E}[\max(\theta_1, \theta_2)].$$

By Lemma 16(iii), the first best requires full revelation by both agents and an F convex on $[\underline{\theta}, \overline{\theta})$ with no atom below $\overline{\theta}$, so F has a density on $[\underline{\theta}, \overline{\theta})$; then also $|d|\big(1 - v(|d|)\big) = 0$ almost surely. As $0 < |d| < \varepsilon$ with positive probability for every $\varepsilon > 0$, $v(x) = 1$ for

arbitrarily small $x > 0$, i.e., $v(x)$ is the noiseless rule, as it is increasing and $v(x) \leq 1$.

C.8.5 Proof of Proposition 11

Lemma 17. *Let F be any prior.*

(i) Every nondecreasing V that satisfies the condition of Lemma 15 equals V_F on $\operatorname{supp}(F)$ *for a rule nondecreasing in the own posterior mean. One such rule is a lottery over coarse gradings: the firm draws a partition of $[\underline{\theta}, \overline{\theta}]$ into intervals, selects the agent whose posterior mean lies in the higher interval, and tosses a coin if both lie in the same one; a posterior mean at an atom of F draws its quantile uniformly from the atom's quantile interval, so the partition may split an atom.*

(ii) In particular, whenever $F \leq_{cx} \bar{U}$, the bound of Proposition 11 is attained by such a rule.

Proof. (i) In quantiles $q = F(x)$, a nondecreasing V that satisfies Lemma 15 is a nondecreasing $v(q) = V(F^{-1}(q))$ majorized by the identity, the interim rule of the noiseless comparison. By Kleiner et al. (2021, Theorems 1 and 3), the extreme points of this set equal the identity outside a collection of disjoint intervals and are constant inside each, and each is implemented by selecting the higher quantile across intervals and randomizing uniformly within an interval; this rule is nondecreasing in the own posterior mean, and so is every mixture of such rules. If F has an atom at x, v is constant on its quantile interval $[F(x^-), F(x)]$, and drawing the quantile of posterior mean x uniformly from that interval, independently across agents, gives posterior mean x the average of v over the interval, which is $V(x)$; the drawn quantile is stochastically increasing in the posterior mean, so the rule stays nondecreasing. (ii) follows from (i), as L is increasing. □

(i) For any prior, full revelation is an equilibrium if there is a convex $\phi \geq V_F$ on $[\underline{\theta}, \overline{\theta}]$ with $\phi = V_F$ on $\operatorname{supp}(F)$, by Jensen's inequality, and, when V_F is upper semicontinuous,

only if (Lemma 13, which extends the necessity part of Dworczak and Martini, 2019, Theorem 2).

As full revelation is a best reply and V_F is upper semicontinuous, Lemma 13 gives a bounded convex $\phi \geq V_F$ on $[\underline{\theta}, \overline{\theta}]$ with $\phi = V_F$ on $\text{supp}(F)$. Let $g := \phi - L$. It is convex, $g(\overline{\theta}) \leq 0$ as $\phi(\overline{\theta}) = V_F(\overline{\theta}) \leq 1 = L(\overline{\theta})$, and $\int g\, dF = \frac{1}{2} - \frac{1}{2} = 0$. If $g(x_1) \leq 0$, then $g \leq \max(g(x_1), g(\overline{\theta})) \leq 0$ on $[x_1, \overline{\theta}]$, so $g > 0$ exactly on some $[\underline{\theta}, x^*)$ and $g \leq 0$ on $[x^*, \overline{\theta}]$. Hence $\int x\, g\, dF = \int (x - x^*) g\, dF \leq 0$, with equality only if $g = 0$ F-almost everywhere, and by Lemma 14, $u_F = p + 2\,\text{Cov}_F(\theta, \phi(\theta)) \leq p + 2\,\text{Cov}_F(\theta, L(\theta)) = p + \frac{\text{Var}_F(\theta)}{\overline{\theta} - p}$. By Lemma 15, L is implementable if and only if $L(\theta) \leq_{cx} U[0,1]$, i.e., $F \leq_{cx} L^{-1}(U[0,1]) = \bar{U}$. The implementing kernel of Lemma 15, written in units of posterior means, is a mean-preserving spread of each posterior mean whose scores under full revelation are distributed as $\bar{U}$; Lemma 17 gives a monotone implementation.

Every concave prior satisfies $F \leq_{cx} \bar{U}$: it lies above the uniform CDF on $[\underline{\theta}, \overline{\theta}]$, so $p \leq \frac{\underline{\theta} + \overline{\theta}}{2}$, and $F - \bar{U}$ is concave on $[\underline{\theta}, \overline{\theta}]$, vanishes at $\overline{\theta}$ and is non-positive below $\underline{\theta}$, so it crosses zero once, from below. A convex prior other than the uniform has $p > \frac{\underline{\theta} + \overline{\theta}}{2}$, so $2p - \overline{\theta} > \underline{\theta}$, and F puts mass below the support of $\bar{U}$.

(ii) is proved in Appendix C.8.11, after the proof of Theorem 3, whose Step 1 it uses.

C.8.6 Proof of Corollary 3

Let $c = 2p - \underline{\theta}$ and $G = U[\underline{\theta}, c]$. As F is concave, it lies above the uniform CDF on $[\underline{\theta}, \overline{\theta}]$, so $p \leq \frac{\underline{\theta} + \overline{\theta}}{2}$, i.e. $c \leq \overline{\theta}$. Moreover, $F - G$ is concave on $[\underline{\theta}, c]$ and non-negative at $\underline{\theta}$, while $F - G = F - 1 \leq 0$ on $[c, \overline{\theta}]$, so $F - G$ changes sign at most once, from positive to negative. Since F and G have the same mean, G is a mean-preserving contraction of F by the single-crossing condition, i.e. it is feasible. Against G, $\overline{v}(s) = G(s)$, which is concave on $[\underline{\theta}, \overline{\theta}]$, so by Jensen's inequality every feasible G' gives $\mathbb{E}_{G'}[G] \leq G(p) = \frac{1}{2} = \mathbb{E}_G[G]$,

and (G, G) is an equilibrium. Finally,

$$u_F = \mathbb{E}[\max(s_1, s_2)] = \underline{\theta} + \frac{2}{3}(c - \underline{\theta}) = \frac{4p - \underline{\theta}}{3}.$$

For uniqueness, let (G, G') be an equilibrium, so that (G', G') and (G', G) are equilibria too. As in Appendix C.8.3, G' has no atom in $(\underline{\theta}, \overline{\theta}]$, and an atom $a > 0$ at $\underline{\theta}$ would not be a best reply to G' either: pooling it with a small mass ε of G' above $\underline{\theta}$ raises the payoff per unit of the atom from $\frac{a}{2}$ to at least a, at a total cost of at most $\varepsilon < \frac{a^2}{2}$. So G' is atomless. As G' is a best reply to G and $G(s) = \min\{\frac{s-\underline{\theta}}{c-\underline{\theta}}, 1\}$ is concave, $\mathbb{E}_{G'}[G] \leq G(p) = \frac{1}{2}$ with equality only if $\operatorname{supp}(G') \subseteq [\underline{\theta}, c]$. As G is a best reply to G' and every mean-preserving contraction of G is feasible, pooling the posterior means of G in any interval $J \subseteq [\underline{\theta}, c]$ does not pay, so G' is convex on $[\underline{\theta}, c]$ by Lemma 12 with $K = G$. Hence $G' \leq G$ on $[\underline{\theta}, c]$, as G is the secant of G' there, and $\int_{\underline{\theta}}^{c} (G - G')dx = \mathbb{E}_{G'}[s] - \mathbb{E}_G[s] = 0$, so $G' = G$.

C.8.7 Proof of Proposition 12

The rule. The greatest common mean-preserving contraction $G^*_{\bar{s}}$ of F and $\bar{U}_{\bar{s}}$ is defined by $I_{G^*_{\bar{s}}}$ being the convex minorant of $\min(I_F, I_{\bar{U}_{\bar{s}}})$. For $\bar{s} \geq 2p - \underline{\theta}$, the uniformizing rule of width $\bar{s}$ scores an agent with posterior mean x by an independent draw S_x with mean x: a draw from a martingale kernel that spreads $G^*_{\bar{s}}$ into $\bar{U}_{\bar{s}}$ if $x \in \operatorname{supp}(G^*_{\bar{s}})$, and $S_x = x$ otherwise; the higher score is selected. For every prior, Lemma 17(i), applied with $G^*_{\bar{s}}$ in place of F to the nondecreasing $L_{\bar{s}}$, which satisfies the condition of Lemma 15 as $G^*_{\bar{s}} \leq_{cx} \bar{U}_{\bar{s}}$, gives a lottery over coarse gradings, nondecreasing in the own posterior mean, under which the value function against $G^*_{\bar{s}}$ equals $L_{\bar{s}}$ on $\operatorname{supp}(G^*_{\bar{s}})$. Off the support, a posterior mean is graded with the highest support point below it, so its value is at most the certificate $L_{\bar{s}}$, because the certificate is nondecreasing, and the bound by $\int L_{\bar{s}}\, dG' = \frac{1}{2}$ below applies to this rule too.

Equilibrium and payoff. Against $G^*_{\bar{s}}$, the rival's score is distributed as $\bar{U}_{\bar{s}}$, whose

CDF equals $L_{\bar{s}}$ on $[2p - \bar{s}, \bar{s}]$. For $x \leq \bar{s}$, S_x lies in this interval, as $\underline{\theta} \geq 2p - \bar{s}$, so the agent wins with probability $\mathbb{E}[L_{\bar{s}}(S_x)] = L_{\bar{s}}(x)$; for $x > \bar{s}$, he wins with probability $1 < L_{\bar{s}}(x)$. Hence every feasible G' yields at most $\int L_{\bar{s}}\, dG' = \frac{1}{2}$, which $G^*_{\bar{s}}$ attains, and $u_F(\bar{s})$ follows from Lemma 14.

Shape. Let $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$ and let $\{x \in [\underline{\theta}, \bar{s}] : F(x) > L_{\bar{s}}(x)\}$ be an interval. For concave F it is, as $F - L_{\bar{s}}$ is then concave on $[\underline{\theta}, \bar{s}]$. Let $D := I_F - I_{\bar{U}_{\bar{s}}}$. Then $D' = F - \bar{U}_{\bar{s}}$ is non-positive below $\underline{\theta}$ and above $\bar{s}$, and on $[\underline{\theta}, \bar{s}]$, where $\bar{U}_{\bar{s}} = L_{\bar{s}}$ as $2p - \bar{s} < \underline{\theta}$, it is positive exactly on an interval. As D vanishes at $\pm\infty$, $D(\underline{\theta}) = -I_{\bar{U}_{\bar{s}}}(\underline{\theta}) < 0$ and $D > 0$ on $(\bar{s}, \overline{\theta})$, D changes sign once, from negative to positive, at some x^*. So $\min(I_F, I_{\bar{U}_{\bar{s}}})$ is convex on each side of x^*, and every interval on which its convex minorant lies strictly below it contains x^*, as on such an interval the minorant is affine and touches the minimum at both ends. Hence the minorant bridges the crossing by a single common supporting line, touching I_F at t_1 (which may be the kink of I_F at $\underline{\theta}$ if F has an atom there) and $I_{\bar{U}_{\bar{s}}}$ at t_2. For concave F, at the ends, $\bar{U}_{2p-\underline{\theta}} \leq_{cx} F$ (proof of Corollary 3) and $F \leq_{cx} \bar{U}_{\overline{\theta}}$ (Appendix C.8.5), so $G^*_{2p-\underline{\theta}} = \bar{U}_{2p-\underline{\theta}}$ and $G^*_{\overline{\theta}} = F$.

C.8.8 Proof of Theorem 3

The bound holds equilibrium by equilibrium:

Lemma 18. *Let $U[\underline{\theta}, 2p - \underline{\theta}] \leq_{cx} F \leq_{cx} \bar{U}$, let the set $\{x \in [\underline{\theta}, \bar{s}'] : F(x) > L_{\bar{s}'}(x)\}$ be an interval for every $\bar{s}' \in (2p - \underline{\theta}, \overline{\theta})$, and let (G, G) be a symmetric equilibrium of a general selection rule in which V_G is nondecreasing on* $\mathrm{supp}(G)$ *and* $\mathrm{Cov}_G(s, V_G(s)) > 0$. *Let $\beta := \frac{\mathrm{Cov}_G(s, V_G(s))}{\mathrm{Var}_G(V_G(s))}$ and $\bar{s} := p + \frac{\beta}{2}$. Then $u_F \leq p + \frac{\mathrm{Var}(G^*_{\bar{s}})}{\bar{s} - p}$, which equals $u_F(\bar{s})$ if $\bar{s} \in [2p - \underline{\theta}, \overline{\theta}]$, and is at most $u_F(2p - \underline{\theta})$ if $\bar{s} < 2p - \underline{\theta}$ and at most $u_F(\overline{\theta})$ if $\bar{s} > \overline{\theta}$.*

Steps 1, 2 and 4–6 below prove Lemma 18; Step 3 reduces the bound of Theorem 3 for every symmetric equilibrium to it, and Step 7 proves that the maximum is attained only at interior widths under the strict inequalities stated after the theorem.

Let (G, G) be a symmetric equilibrium, $s \sim G$ and $V := V_G$. By Lemma 14, $u_F = p + 2\,\mathrm{Cov}(s, V(s))$. The stated maximum exists, as $\mathrm{Var}(G^*_{\bar{s}})$ is continuous in $\bar{s}$.

Step 1 (convexity on the support). By Lebesgue's differentiation theorem for G, the set $A \subseteq \mathrm{supp}(G)$ of points at which V is approximately continuous with respect to G has $G(A) = 1$. For $s_1 < s_2 < s_3$ in A, run the chord argument of the proof of Lemma 13 with G in place of F, with the two neighbourhoods replaced by sets of positive G-mass within δ of s_1 and s_3 on which V is within ε of $V(s_1)$ and $V(s_3)$, and with the target replaced by a point x within δ of s_2 with $|V(x) - V(s_2)| \leq \varepsilon$; as every contraction of G is feasible, letting $\varepsilon, \delta \to 0$ gives $V(s_2) \leq \frac{s_3 - s_2}{s_3 - s_1} V(s_1) + \frac{s_2 - s_1}{s_3 - s_1} V(s_3)$. So V is the restriction to A of a convex function φ on the convex hull of A, and $V(s) = \varphi(s)$ almost surely.

Step 2 (implementability). As in the proof of Lemma 15, with G in place of F, $V(s) \leq_{cx} U[0, 1]$.

Step 3 (monotone pooling). Let Q be the quantile function of G and $r(t) := \varphi(Q(t))$, which is quasi-convex on $[0, 1]$, as φ is convex and Q nondecreasing. Let q^* be the largest minimizer of the running average $c(q) := \frac{1}{q} \int_0^q r$ on $[0, 1]$, with $c(0) := r(0^+)$, and $c^* := c(q^*)$. Minimality of c^* and quasi-convexity of r, whose sublevel sets are intervals, give three facts: $r \geq c^*$ almost everywhere on $(q^*, 1]$, as otherwise c would fall below c^* to the right of q^*; r is nondecreasing there; and on $[0, q^*]$, $r - c^*$ is positive below some j and nonpositive above it, as otherwise c would fall below c^* just left of q^*. Now pool the lowest q^* quantiles into one realization, their mean μ, with winning probability c^*. The new signal is a contraction of G, so it is still $\leq_{cx} F$; the new winning probability is a conditional expectation of $V(s)$, so it is still $\leq_{cx} U[0, 1]$ by Step 2; and $\mathbb{E}[(s - p)V(s)]$ rises by $-\int_0^{q^*} (Q(t) - \mu)(r(t) - c^*)\, dt = -\int_0^{q^*} (Q(t) - Q(j))(r(t) - c^*)\, dt \geq 0$, so u_F weakly rises. The new value function is the restriction of $\max(\varphi, c^*)$ to $[\mu, \infty)$, which is convex and nondecreasing, as $\varphi(\mu) \leq c^*$ by Jensen's inequality. As Steps 4–6 use only these properties, assume from now on that φ is nondecreasing and that $\mathrm{Cov}(s, V(s)) > 0$,

as otherwise $u_F \le p$, below every $u_F(\bar{s})$. This proves the claim in Subsection 4.4 that monotonicity in the own posterior mean is without loss for the firm.

Step 4 (the width). Let $\beta := \frac{\mathrm{Cov}(s,V(s))}{\mathrm{Var}(V(s))}$, $\bar{s} := p + \frac{\beta}{2}$, $\psi := p + \beta(\varphi - \frac{1}{2})$, which is convex and nondecreasing, and $Y := \psi(s)$. By Step 2, $Y \le_{cx} U[p - \frac{\beta}{2}, p + \frac{\beta}{2}] = \bar{U}_{\bar{s}}$, so $\mathbb{E}Y = p$; by the choice of β, $\mathrm{Cov}(Y, s - Y) = \beta \,\mathrm{Cov}(V, s) - \beta^2 \mathrm{Var}(V) = 0$; and $\mathbb{E}[(Y - p)(s - p)] = 2(\bar{s} - p)\,\mathrm{Cov}(s, V(s))$. So it suffices to show that $\mathbb{E}[(Y - p)(s - p)] \le \mathrm{Var}(G^*_{\bar{s}})$.

Step 5 (a dual bound). Suppose that w_1, w_2 are convex, $\int w_1\, dF + \int w_2\, d\bar{U}_{\bar{s}} = \mathrm{Var}(G^*_{\bar{s}})$, a is concave, and

$$R(s, y) := w_1(s) + w_2(y) - (y - p)(s - p) - a(y)(s - y) \ge 0 \qquad \text{for all } s, y.$$

As $G \le_{cx} F$ and $Y \le_{cx} \bar{U}_{\bar{s}}$,

$$\mathbb{E}[(Y-p)(s-p)] = \mathbb{E}w_1(s) + \mathbb{E}w_2(Y) - \mathbb{E}[a(Y)(s-Y)] - \mathbb{E}R(s,Y) \le \mathrm{Var}(G^*_{\bar{s}}) - \mathbb{E}[a(Y)(s-Y)].$$

Such functions exist. For $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, let t_1, t_2 be as in Proposition 12, which applies by the hypothesis on $\{F > L_{\bar{s}}\}$, let C be the chord of $(x - p)^2$ over $[t_1, t_2]$, and let

$$w_1(x) := \begin{cases} (x-p)^2, & x \le t_1, \\ C(x), & x \ge t_1, \end{cases} \qquad w_2(x) := \begin{cases} 0, & x \le t_2, \\ (x-p)^2 - C(x), & x \ge t_2, \end{cases}$$

$$a(y) := \min\{y - p,\ t_1 - p,\ t_1 + t_2 - p - y\}.$$

Here w_1 is affine where $I_{G^*} < I_F$ (above t_1), so $\int w_1\, dF = \int w_1\, dG^*$; w_2 is affine where $I_{G^*} < I_{\bar{U}_{\bar{s}}}$ (below t_2), so $\int w_2\, d\bar{U}_{\bar{s}} = \int w_2\, dG^*$; and $w_1 + w_2 = (x - p)^2$ on $\mathrm{supp}(G^*) \subseteq [\underline{\theta}, t_1] \cup [t_2, \bar{s}]$, so the two integrals sum to $\mathrm{Var}(G^*_{\bar{s}})$. For $y \le t_1$, $R(s, y)$ is the gap between w_1 and its tangent at y; for $y \ge t_2$, $R(s, y) = w_1(s) - C(s) \ge 0$; and for $t_1 < y < t_2$, $R(s, y) = (s - t_1)(t_2 - y)$ if $s \ge t_1$ and $R(s, y) = (s - t_1)(s - y)$ if

$s \le t_1$, both nonnegative. For $\bar{s} \le 2p - \underline{\theta}$, take $w_1 = 0$, $w_2 = (x-p)^2$ and $a(y) = p - y$, so that $R = 0$ and the value is $\mathrm{Var}(\bar{U}_{\bar{s}}) = \mathrm{Var}(G^*_{\bar{s}})$, as $\bar{U}_{\bar{s}} \le_{cx} \bar{U}_{2p-\underline{\theta}} \le_{cx} F$; for $\bar{s} \ge \overline{\theta}$, take $w_1 = (x-p)^2$, $w_2 = 0$ and $a(y) = y - p$, so that $R = (s-y)^2$ and the value is $\mathrm{Var}(F) = \mathrm{Var}(G^*_{\bar{s}})$, as $F \le_{cx} \bar{U}_{\overline{\theta}} \le_{cx} \bar{U}_{\bar{s}}$. In these two cases, $p + \mathrm{Var}(G^*_{\bar{s}})/(\bar{s} - p)$ equals $p + \frac{\bar{s}-p}{3}$, which is increasing, and $p + \frac{\mathrm{Var}(F)}{\bar{s}-p}$, which is decreasing, so a width outside $[2p - \underline{\theta}, \overline{\theta}]$ is weakly beaten by the nearest end of the family.

Step 6 (the sign). Let $\nu(B) := \mathbb{E}[(s - Y)\mathbb{1}\{Y \in B\}]$, so that $\nu(\mathbb{R}) = 0$ and $\int y \, d\nu = \mathrm{Cov}(Y, s - Y) = 0$. As $x - \psi(x)$ is concave, the posterior means with $s > \psi(s)$ lie in an interval J that contains every other posterior mean between two of them. If J contains no posterior mean, $\nu = 0$. Otherwise, let (y_a, y_b) be the interior of the convex hull of the values $\psi(s)$ of posterior means $s \in J$. If $\psi(s) \in (y_a, y_b)$, then $\psi(s') < \psi(s) < \psi(s'')$ for some posterior means $s', s'' \in J$, so $s' < s < s''$ as ψ is nondecreasing, and $s \in J$. Hence $\nu \ge 0$ on (y_a, y_b) and $\nu \le 0$ outside $[y_a, y_b]$. With ℓ the chord of a over $[y_a, y_b]$ (a supporting line if $y_a = y_b$), $a - \ell$ is nonnegative on $[y_a, y_b]$ and nonpositive outside it, so $\mathbb{E}[a(Y)(s - Y)] = \int (a - \ell)\, d\nu \ge 0$. With Step 5, $\mathbb{E}[(Y - p)(s - p)] \le \mathrm{Var}(G^*_{\bar{s}})$, which proves the bound.

Step 7 (interior widths). Let $I_{\bar{U}_{2p-\underline{\theta}}} < I_F < I_{\bar{U}_{\overline{\theta}}}$ on $(\underline{\theta}, \overline{\theta})$. Every concave F that is not uniform satisfies this: then $p < \frac{\underline{\theta}+\overline{\theta}}{2}$, $F \le_{cx} \bar{U}_{\overline{\theta}}$ and $\bar{U}_{2p-\underline{\theta}} \le_{cx} F$, and both inequalities are strict on $(\underline{\theta}, \overline{\theta})$, since in both cases the CDFs cross once. The strict inequalities give $2p - \underline{\theta} < \overline{\theta}$, as otherwise $\bar{U}_{2p-\underline{\theta}} = \bar{U}_{\overline{\theta}}$, so $p < \frac{\underline{\theta}+\overline{\theta}}{2}$ and $I_{\bar{U}_{\overline{\theta}}}(\underline{\theta}) > 0 = I_F(\underline{\theta})$. Two facts are used. For distributions with mean p, $\mathrm{Var}(K_1) - \mathrm{Var}(K_2) = 2\int (I_{K_1} - I_{K_2})\, dx$. For a uniform distribution of half-width r around p, $\frac{\partial I}{\partial r} = \frac{(x-p+r)(r-x+p)}{4r^2}$ on $[p - r, p + r]$ and 0 elsewhere.

Near the full revelation end, let $\eta := \overline{\theta} - \bar{s}$ and $h := \max_x (I_F - I_{\bar{U}_{\bar{s}}})^+$. Then $\max(I_F - h,\ x - p,\ 0)$ is a convex minorant of $\min(I_F, I_{\bar{U}_{\bar{s}}})$, so $\mathrm{Var}(F) - \mathrm{Var}(G^*_{\bar{s}}) \le 2(\overline{\theta} - \underline{\theta})h$. Write $I_F - I_{\bar{U}_{\bar{s}}} = -(I_{\bar{U}_{\overline{\theta}}} - I_F) + (I_{\bar{U}_{\overline{\theta}}} - I_{\bar{U}_{\bar{s}}})$. The derivative formula, with $r - x + p \le \overline{\theta} - x$

and $x - p + r \le 2r$, bounds the second term by $\frac{\eta(\overline{\theta}-x)^+}{2(\bar{s}-p)}$. Fix $\delta > 0$: on $[\underline{\theta}, \overline{\theta} - \delta]$ the first term is bounded away from zero, so there is no violation for small η; on $[\overline{\theta} - \delta, \overline{\theta}]$ the violation is at most $\frac{\eta\delta}{2(\bar{s}-p)}$. Hence $h = o(\eta)$, and

$$u_F(\bar{s}) - u_F(\overline{\theta}) \ge \frac{\mathrm{Var}(F)\,\eta}{(\bar{s}-p)(\overline{\theta}-p)} - \frac{2(\overline{\theta}-\underline{\theta})\,o(\eta)}{\bar{s}-p} > 0 \quad \text{for small } \eta.$$

Near the noiseless end, let $\bar{s}_0 := 2p - \underline{\theta}$, $\eta := \bar{s} - \bar{s}_0$, and $h' := \max_x (I_{\bar{U}_{\bar{s}}} - I_F)^+$. Symmetrically, $\max(I_{\bar{U}_{\bar{s}}} - h', x - p, 0)$ is a convex minorant of the minimum, so $\mathrm{Var}(\bar{U}_{\bar{s}}) - \mathrm{Var}(G^*_{\bar{s}}) \le 4(\bar{s}-p)h'$. Write $I_{\bar{U}_{\bar{s}}} - I_F = (I_{\bar{U}_{\bar{s}}} - I_{\bar{U}_{\bar{s}_0}}) - (I_F - I_{\bar{U}_{\bar{s}_0}})$. The derivative formula, with $r - x + p \le 2r$ and $x - p + r \le x - \underline{\theta} + \eta$, bounds the first term by $\frac{\eta(x-\underline{\theta}+\eta)^+}{2(p-\underline{\theta})}$; below $\underline{\theta}$, the violation is at most $I_{\bar{U}_{\bar{s}}}(\underline{\theta}) = O(\eta^2)$; and near $\overline{\theta}$ the first term vanishes for $x \ge \bar{s}$, while on $[\underline{\theta} + \delta, \bar{s}]$, a compact subset of $(\underline{\theta}, \overline{\theta})$ as $\bar{s}_0 < \overline{\theta}$, the second is bounded away from zero and the first is at most $\frac{\eta(\overline{\theta}-\underline{\theta}+\eta)}{2(p-\underline{\theta})} \to 0$. So $h' = o(\eta)$. As $\mathrm{Var}(\bar{U}_{\bar{s}}) = \frac{(\bar{s}-p)^2}{3}$ and $G^*_{\bar{s}_0} = \bar{U}_{\bar{s}_0}$,

$$u_F(\bar{s}) - u_F(\bar{s}_0) \ge \frac{\bar{s}-p}{3} - \frac{\bar{s}_0-p}{3} - 4h' = \frac{\eta}{3} - o(\eta) > 0 \quad \text{for small } \eta.$$

So neither end maximizes $u_F(\bar{s})$, and the maximum is attained only at interior widths.

C.8.9 Single-peaked and U-shaped priors

Lemma 19. *Let F have a density f that, for some m, is either nondecreasing on $[\underline{\theta}, m]$ and nonincreasing on $[m, \overline{\theta}]$ (*single-peaked*) or nonincreasing on $[\underline{\theta}, m]$ and nondecreasing on $[m, \overline{\theta}]$ (*U-shaped*).*

(i) If F is single-peaked, then $F \le_{cx} \bar{U}$ if and only if $p \le \frac{\underline{\theta}+\overline{\theta}}{2}$.

(ii) For every $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, the set $\{x \in [\underline{\theta}, \bar{s}] : F(x) > L_{\bar{s}}(x)\}$ is an interval, i.e., F satisfies condition (iii) of Theorem 3.

Proof. (i) If $F \leq_{cx} \bar{U}$, the support of F lies in that of $\bar{U}$, so $\underline{\theta} \geq 2p - \overline{\theta}$. Conversely let $p \leq \frac{\underline{\theta}+\overline{\theta}}{2}$ and $g := L - F$ on $[\underline{\theta}, \overline{\theta}]$. Then $g(\underline{\theta}) = L(\underline{\theta}) = \frac{1}{2} - \frac{p-\underline{\theta}}{2(\overline{\theta}-p)} \geq 0$, $g(\overline{\theta}) = 0$, and g is concave on $[\underline{\theta}, m]$ and convex on $[m, \overline{\theta}]$. On $[\underline{\theta}, m]$ the superlevel set $\{g \geq 0\}$ is an interval containing $\underline{\theta}$; on $[m, \overline{\theta}]$ the sublevel set $\{g \leq 0\}$ is an interval containing $\overline{\theta}$. So g is first ≥ 0 and then ≤ 0 on $[\underline{\theta}, \overline{\theta}]$ (if $g(m) > 0$ the first interval is all of $[\underline{\theta}, m]$; if $g(m) \leq 0$ the second is all of $[m, \overline{\theta}]$). As $2p - \overline{\theta} \leq \underline{\theta}$, $\bar{U} = L$ on $[\underline{\theta}, \overline{\theta}]$, so $D := F - \bar{U}$ equals $-g$ there, is non-positive below $\underline{\theta}$ and vanishes above $\overline{\theta}$: it changes sign once, from $-$ to $+$. As F and $\bar{U}$ have the same mean, $I_F - I_{\bar{U}} = \int_{-\infty}^{\cdot} D\, dt$ vanishes at $\pm\infty$, first decreases and then increases, so $I_F \leq I_{\bar{U}}$, i.e. $F \leq_{cx} \bar{U}$.

(ii) Let $\bar{s} \in (2p - \underline{\theta}, \overline{\theta})$, $d := \frac{1}{2(\bar{s}-p)}$, the slope of $L_{\bar{s}}$, and $h := F - L_{\bar{s}}$ on $[\underline{\theta}, \bar{s}]$, so that $h(y) - h(x) = \int_x^y (f - d)\, dt$ for $x < y$. If F is single-peaked, $f - d$ is nondecreasing and then nonincreasing, so $J := \{x \in [\underline{\theta}, \bar{s}] : f(x) > d\}$ is an interval, and h is nonincreasing on each of the at most two components of $[\underline{\theta}, \bar{s}] \setminus J$ and nondecreasing on J. As $\bar{s} > 2p - \underline{\theta}$, $h(\underline{\theta}) = -L_{\bar{s}}(\underline{\theta}) = \frac{p-\underline{\theta}}{2(\bar{s}-p)} - \frac{1}{2} < 0$, so $h < 0$ on $[\underline{\theta}, \inf J]$ (on all of $[\underline{\theta}, \bar{s}]$ if $J = \emptyset$); $\{h > 0\} \cap J$ is a final segment of J; and on $[\sup J, \bar{s}]$, $\{h > 0\}$ is an initial segment, nonempty only if $h(\sup J) > 0$. So $\{h > 0\}$ is an interval. If F is U-shaped, $f - d$ is nonincreasing and then nondecreasing, so $J' := \{x \in [\underline{\theta}, \bar{s}] : f(x) < d\}$ is an interval, and h is nondecreasing before J', nonincreasing on J' and nondecreasing after it. As $\bar{s} < \overline{\theta} = \max \mathrm{supp}(F)$, $h(\bar{s}) = F(\bar{s}) - 1 < 0$, so $h < 0$ on $[\sup J', \bar{s}]$, and on $[\underline{\theta}, \sup J']$, where h is nondecreasing and then nonincreasing, $\{h > 0\}$ is an interval. □

C.8.10 Proof of Corollary 4

Let $\underline{\theta}'$ and $\overline{\theta}'$ be the bounds of the support of F'. As $F \leq_{cx} F'$, $\underline{\theta}' \leq \underline{\theta}$ and $\overline{\theta}' \geq \overline{\theta}$, so every width in $[2p - \underline{\theta}, \overline{\theta}]$ that is at least $2p - \underline{\theta}'$ is available under F'. For a smaller width $\bar{s}$, $I_{G^*_{\bar{s}}} \leq I_{\bar{U}_{\bar{s}}}$ gives $G^*_{\bar{s}} \leq_{cx} \bar{U}_{\bar{s}}$, so $u_F(\bar{s}) \leq p + \frac{\bar{s}-p}{3} < p + \frac{p-\underline{\theta}'}{3} = u_{F'}(2p - \underline{\theta}')$, as $\bar{U}_{2p-\underline{\theta}'} \leq_{cx} F'$ by (i). For each available width $\bar{s}$, $I_F \leq I_{F'}$ gives $\min(I_F, I_{\bar{U}_{\bar{s}}}) \leq \min(I_{F'}, I_{\bar{U}_{\bar{s}}})$, and hence

the same order for their convex minorants, $I_{G^*_{\bar{s}}} \leq I_{G'^*_{\bar{s}}}$. Both have mean p, so $G^*_{\bar{s}} \leq_{cx} G'^*_{\bar{s}}$ and $\mathrm{Var}(G^*_{\bar{s}}) \leq \mathrm{Var}(G'^*_{\bar{s}})$, i.e., $u_F(\bar{s}) \leq u_{F'}(\bar{s})$ by Proposition 12. By Theorem 3 and Proposition 12, the second best under F is the maximum of $u_F(\bar{s})$ over $[2p - \underline{\theta}, \overline{\theta}]$, which is thus at most the second best under F'.

C.8.11 Proof of Proposition 11(ii)

Let $V := V_G$ and $L_G(x) := \frac{1}{2} + \frac{x-p}{2(\bar{s}_G - p)}$, so $L_G(\bar{s}_G) = 1$ and $\int L_G \, dG = \frac{1}{2}$. By Step 1 of the proof of Theorem 3, which uses only that every contraction of G is feasible, there is a set A with $G(A) = 1$ and a convex φ on the convex hull of A with $V = \varphi$ on A. Let $g := \varphi - L_G$. As $\bar{s}_G \in \mathrm{supp}(G)$, A has points arbitrarily close to $\bar{s}_G$, where $g \leq 1 - L_G \to 0$; so the limit of the convex g at $\bar{s}_G$, which exists, is at most 0, and if G has an atom at $\bar{s}_G$, then $\bar{s}_G \in A$ and $g(\bar{s}_G) \leq 0$. As in (i), $\{g > 0\}$ is an initial segment ending at some x^*, so $(s - x^*)g(s) \leq 0$ on A, and with $\int g \, dG = \frac{1}{2} - \frac{1}{2} = 0$, $\int s \, g \, dG \leq 0$. By Lemma 14, as $\mathbb{E}_G[s] = p$ and $\mathbb{E}_G[V] = \frac{1}{2}$, $u_F = 2\int sV \, dG \leq 2\int sL_G \, dG = p + \frac{\mathrm{Var}_G(s)}{\bar{s}_G - p}$. If $\bar{s}_G = \overline{\theta}$, then $\mathrm{Var}_G(s) \leq \mathrm{Var}_F(\theta)$ as $G \leq_{cx} F$.

C.8.12 The noiseless equilibrium for a general prior

For priors with a positive, continuously differentiable density with finitely many peaks, the following structure is due to Hwang et al. (2026, Theorem 1). I prove it under the weaker conditions used in Subsection 5.2.4, as the proof of Theorem 4 uses the details of the argument.

Lemma 20. *Let F be atomless, with $\mathrm{supp}(F) = [\underline{\theta}, \overline{\theta}]$ and a density f that is continuous on $(\underline{\theta}, \overline{\theta})$. A symmetric profile (G, G) is an equilibrium of the noiseless rule if and only if*

(a) G is atomless and $\mathrm{supp}(G) = [\underline{\theta}, c]$ for some $c \in (\underline{\theta}, \overline{\theta}]$;

(b) G is convex on $[\underline{\theta}, c]$;

(c) $G \leq_{cx} F$;

(d) G *is affine on every component of* $(\underline{\theta}, c) \setminus E$, *where* $E := \{x \in [\underline{\theta}, c] : I_G(x) = I_F(x)\}$.

In that case, $G = F$ *on* E; $c \notin E$ *if* $c < \overline{\theta}$; *the function* ψ *that equals* G *on* $[\underline{\theta}, c]$ *and* $1 + G'(c^-)(x - c)$ *on* $[c, \overline{\theta}]$ *is convex,* $\psi \geq G$, *and* $d\psi'$ *is carried by* E; *the firm's utility is* $\mathbb{E}_G[\max(s_1, s_2)]$; *and* (G, G) *is the unique symmetric equilibrium.*

Proof. Against a rival who plays K, the noiseless rule gives a posterior mean x the payoff $\bar{K}(x) = K(x^-) + \frac{1}{2}K(\{x\})$, and G is a best reply to itself if and only if it maximizes $\int \bar{G}\, dG'$ over feasible G'.

No atoms. Let (G, G) be an equilibrium. An atom at $\underline{\theta}$ or at $\overline{\theta}$ is infeasible, as in Appendix C.8.3: at $\underline{\theta}$ because $I_F(x) = o(x - \underline{\theta})$, and at $\overline{\theta}$ because $I_F - I_G \geq 0$ vanishes there, so its left derivative $F(\overline{\theta}-) - G(\overline{\theta}-)$ is not positive. An interior atom is ruled out by the deviation of Appendix C.8.3. So $\bar{G} = G$ is continuous.

The certificate. For a continuous convex ψ on $[\underline{\theta}, \overline{\theta}]$ with $\psi'(\underline{\theta}^+)$ finite and distributions K_1, K_2 on $[\underline{\theta}, \overline{\theta}]$ with equal means, writing $\psi(x) = \psi(\underline{\theta}) + \psi'(\underline{\theta}^+)(x - \underline{\theta}) + \int (x - a)^+\, d\psi'(a)$ (at $x = \overline{\theta}$ by monotone convergence) and using $\int (x-a)^+\, dK(x) = p - a + I_K(a)$ gives

$$\int \psi\, dK_1 - \int \psi\, dK_2 = \int \left(I_{K_1} - I_{K_2}\right) d\psi', \tag{1}$$

which extends by linearity to differences of such functions. Hence, if G is continuous and such a $\psi \geq G$ has $\psi = G$ on $\operatorname{supp}(G)$ and $d\psi'$ carried by $\{I_F = I_G\}$, then $\int G\, dG' \leq \int \psi\, dG' \leq \int \psi\, dF = \int \psi\, dG = \int G\, dG$ for every feasible G', so G is a best reply to itself.

Necessity. Let $\ell := \min \operatorname{supp}(G) < c := \max \operatorname{supp}(G)$. Every $G' \leq_{cx} G$ is feasible, so Lemma 13, applied to the prior G and $V = G$, gives a convex $\phi \geq G$ on $[\ell, c]$ with $\phi = G$ on $\operatorname{supp}(G)$. If $a < b$ are in $\operatorname{supp}(G)$ with $G(a) = G(b)$, then ϕ equals $G(a)$ at a and b and is at least $G \geq G(a)$ between them, so it is constant on $[a, b]$ and nonincreasing

on $[\ell, a]$. Then $0 = G(\ell) = \phi(\ell) \geq \phi(a) = G(b)$, which contradicts $\ell \in \operatorname{supp}(G)$. So $\operatorname{supp}(G) = [\ell, c]$, and $G = \phi$ is convex on $[\ell, c]$; extended by 0, it is convex on $[\underline{\theta}, c]$. Let $D := I_F - I_G \geq 0$, and let $[x_1, x_3] \subseteq [\underline{\theta}, c]$ with $\kappa := \min_{[x_1,x_3]} D > 0$. Then G is affine on $[x_1, x_3]$. Otherwise, being convex, G lies strictly below its secant s through x_1 and x_3 on (x_1, x_3) and is not constant there. So some $[y_1, y_2] \subset (x_1, x_3)$ has G-mass $m > 0$, and $\delta := \min_{[y_1,y_2]}(s - G) > 0$. Splitting a share ε of that mass, each point onto $\{x_1, x_3\}$ with its own mean, gives G_ε with $0 \leq I_{G_\varepsilon} - I_G \leq \varepsilon m(x_3 - x_1)$, and the difference vanishes off $[x_1, x_3]$. So $G_\varepsilon \leq_{cx} F$ for small ε, while $\int G\, dG_\varepsilon - \int G\, dG = \varepsilon \int_{[y_1,y_2]}(s - G)\, dG \geq \varepsilon m\delta > 0$, a contradiction. If $\ell > \underline{\theta}$, then $D = I_F > 0$ on $(\underline{\theta}, \ell]$, so $D > 0$ on some $[x_1, \ell + \eta]$ with $\underline{\theta} < x_1 < \ell < \ell + \eta < c$. But G vanishes on $[x_1, \ell]$ and is positive at $\ell + \eta$, so it is not affine there; hence $\ell = \underline{\theta}$, which proves (a)–(c). Each compact subinterval of a component of $(\underline{\theta}, c) \setminus E$ has $\min D > 0$, which proves (d). If $c < \overline{\theta}$, then $D(x) = \int_x^{\overline{\theta}} (1 - F)\, dt > 0$ on $[c, \overline{\theta})$, so $c \notin E$, and by (d) G is affine on a left neighbourhood of c, so $G'(c^-) < \infty$. At every $x \in E \cap (\underline{\theta}, \overline{\theta})$, D is differentiable and minimal, so $F(x) = G(x)$.

Sufficiency and payoff. Given (a)–(d), the function ψ of the statement is convex, $\psi \geq G$, $\psi = G$ on $[\underline{\theta}, c]$, and $d\psi'$ is carried by E, so (G, G) is an equilibrium. The firm selects the higher posterior mean, so $u_F = \mathbb{E}_G[\max(s_1, s_2)]$.

Uniqueness. Let G and $\hat{G}$ satisfy (a)–(d), with certificates $\psi, \hat{\psi}$, tops $c, \hat{c}$ and contact sets $E, \hat{E}$. By interchangeability $(G, \hat{G})$ is an equilibrium, so $\frac{1}{2} = \int G\, d\hat{G} \leq \int \psi\, d\hat{G} \leq \int \psi\, dF = \frac{1}{2}$. Hence $\psi = G$ on $[\underline{\theta}, \hat{c}]$, and, by (1), $d\psi'$ is carried by $\hat{E}$. If $\hat{c} > c$, then $\psi = 1$ on $[c, \hat{c}]$, so $G'(c^-) = 0$ and $G = 0$ on $[\underline{\theta}, c)$, which is absurd. So $\hat{c} \leq c$, symmetrically $c \leq \hat{c}$, and dG', $d\hat{G}'$ are carried by $E \cap \hat{E}$, where $G = F = \hat{G}$. Off this set, G and $\hat{G}$ are affine on each interval and agree at its endpoints, which lie in $E \cap \hat{E}$ or are $\underline{\theta}$ or c, so $G = \hat{G}$.

□

Lemma 21. *Let F have $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$ and a density f that is continuous on $[\underline{\theta}, \overline{\theta}]$. Then the noiseless rule has a symmetric equilibrium.*

Proof. Positive polynomial densities. Let f first be a polynomial that is positive on $[\underline{\theta}, \overline{\theta}]$. If f is constant, $G := F$ satisfies (a)–(d) of Lemma 20 with $E = [\underline{\theta}, \overline{\theta}]$. Otherwise f' has finitely many zeros, so f has finitely many peaks. Translating types so that $\underline{\theta} > 0$ changes neither the noiseless rule nor the convex order. Then the advertising game of Hwang et al. (2026) with two firms and a price $q \in (0, \underline{\theta})$ has full coverage, and a firm's demand against a rival who plays G is $\int G \, dG'$. By their Theorem 1, it has a symmetric equilibrium (G, G), and G has no atoms (Hwang et al., 2026, Lemma 1 and footnote 22); an atom at $\underline{\theta}$ is infeasible anyway, as $I_F(x) = o(x - \underline{\theta})$. So every deviation $G' \leq_{cx} F$ earns the same payoff under the noiseless rule, and (G, G) is its equilibrium.

Approximation. In general, by the Weierstrass theorem, take polynomials P_k with $|P_k - f| \leq \frac{1}{k}$ on $[\underline{\theta}, \overline{\theta}]$, and let $f_k := (P_k + \frac{2}{k})/Z_k$ with $Z_k := \int_{\underline{\theta}}^{\overline{\theta}} (P_k + \frac{2}{k}) \, dt$. Then $f_k \geq \frac{1}{kZ_k} > 0$, $Z_k \to 1$ and $f_k \to f$ uniformly, so the CDFs F_k of the f_k satisfy $F_k \to F$ and $I_{F_k} \to I_F$ uniformly, and $M := \sup_k \max_{[\underline{\theta}, \overline{\theta}]} f_k < \infty$. By the first part and Lemma 20, the noiseless rule for F_k has a symmetric equilibrium (G_k, G_k), with G_k atomless and convex on $[\underline{\theta}, c_k]$, $c_k := \max \operatorname{supp}(G_k)$, and with contact set E_k.

Lipschitz bound. The left derivative $G_k'(c_k^-)$ is at most M. If c_k is the limit of points $x_n < c_k$ of the closed set E_k, then $c_k \in E_k$, and as $G_k = F_k$ on E_k, $G_k'(c_k^-) = \lim_n \frac{F_k(c_k) - F_k(x_n)}{c_k - x_n} = f_k(c_k) \leq M$. Otherwise G_k is affine, with some slope γ, on a component (a, c_k) of $(\underline{\theta}, c_k) \setminus E_k$, where $a \in E_k$ and possibly $a = \underline{\theta}$. As $I_{G_k} \leq I_{F_k}$ with equality at a, and $G_k(a) = F_k(a)$, for $x \in (a, c_k)$,

$$\gamma \frac{(x-a)^2}{2} = \int_a^x \big(G_k(t) - G_k(a)\big) \, dt \leq \int_a^x \big(F_k(t) - F_k(a)\big) \, dt \leq M \frac{(x-a)^2}{2},$$

so $\gamma \leq M$. As G_k is convex on $[\underline{\theta}, c_k]$ and equals 1 above c_k, it is M-Lipschitz on $[\underline{\theta}, \overline{\theta}]$.

The limit. By the Arzelà–Ascoli theorem, along a subsequence, which I keep, $G_k \to G$ uniformly on $[\underline{\theta}, \overline{\theta}]$. Then G is nondecreasing and M-Lipschitz, with $G(\underline{\theta}) = 0$ and $G(\overline{\theta}) = 1$, so it is an atomless CDF, and $I_{G_k} \to I_G$ uniformly. As $I_{G_k} \leq I_{F_k}$ with equality at $\overline{\theta}$, $I_G \leq I_F$ with equality at $\overline{\theta}$, so $G \leq_{cx} F$.

Deviations. Let $G' \leq_{cx} F$, and take random variables $\theta \sim F$ and $s \sim G'$ with $\mathbb{E}[\theta|s] = s$. Let $\theta_k := F_k^{-1}(F(\theta))$. As F is continuous, $F(\theta)$ is uniform on $[0,1]$, so $\theta_k \sim F_k$. As F is continuous and strictly increasing on $[\underline{\theta}, \overline{\theta}]$, for $y \in (0,1)$, $x := F^{-1}(y)$ and $\eta > 0$, $F(x-\eta) < y < F(x+\eta)$, and these inequalities hold for F_k for large k; so $F_k^{-1} \to F^{-1}$ on $(0,1)$, $\theta_k \to \theta$ almost surely, and $\mathbb{E}|\theta_k - \theta| \to 0$ by bounded convergence. Let $s_k := \mathbb{E}[\theta_k|s]$, with distribution G_k'. Then $G_k' \leq_{cx} F_k$ and $\mathbb{E}|s_k - s| \leq \mathbb{E}|\theta_k - \theta| \to 0$. As (G_k, G_k) is an equilibrium and G_k is atomless, $\mathbb{E}[G_k(s_k)] = \int G_k \, dG_k' \leq \frac{1}{2}$, and $\left|\mathbb{E}[G_k(s_k)] - \mathbb{E}[G(s)]\right| \leq \sup|G_k - G| + M\,\mathbb{E}|s_k - s| \to 0$. Hence $\int G \, dG' \leq \frac{1}{2} = \int G \, dG$, as G is atomless. Against the atomless G, a posterior mean x earns $G(x)$ under the noiseless rule, so G is a best reply to itself, and (G, G) is a symmetric equilibrium. □

Definition 6. Let F be atomless, with $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$ and a density f that is continuous on $(\underline{\theta}, \overline{\theta})$. Its noiseless equilibrium is *well behaved* if

(H1) the noiseless rule has a symmetric equilibrium (G_0, G_0); let $c := \max \operatorname{supp}(G_0)$ and $E_0 := \{x \in [\underline{\theta}, c] : I_{G_0}(x) = I_F(x)\}$;

(H2) $[\underline{\theta}, c]$ is the union of finitely many intervals, each of which is either a *revealed stretch*, a nondegenerate interval contained in E_0 on which F is strictly convex and on whose interior F is twice continuously differentiable except at finitely many points, or a *block*, an interval on which G_0 is affine and which meets E_0 in finitely many points;

(H3) if $x_0 \in E_0$ is an endpoint of a component of $(\underline{\theta}, c) \setminus E_0$ on which G_0 is affine with slope γ, then $f \geq \gamma$ a.e. on a one-sided neighbourhood of x_0 in that component.

(H1) holds whenever f is continuous on $[\underline{\theta}, \overline{\theta}]$ (Lemma 21); the next result gives all three conditions under the assumption of Theorem 4, a density with finitely many turning points.

Proposition 14. *Let F have a density f that is continuous on $[\underline{\theta}, \overline{\theta}]$, and let $\underline{\theta} = z_0 < z_1 < \cdots < z_m = \overline{\theta}$ be such that, for each j, f is strictly monotone on $[z_{j-1}, z_j]$ and continuously differentiable on (z_{j-1}, z_j). Then the noiseless equilibrium of F is well behaved, with at most m pools.*

Proof. (H1). Being strictly monotone on each $[z_{j-1}, z_j]$, f has finitely many zeros, so F is strictly increasing, $\operatorname{supp}(F) = [\underline{\theta}, \overline{\theta}]$, and Lemma 21 gives a symmetric equilibrium (G_0, G_0) of the noiseless rule. Let $c := \max \operatorname{supp}(G_0)$, $E_0 := \{x \in [\underline{\theta}, c] : I_{G_0}(x) = I_F(x)\}$ and $\Delta := I_F - I_{G_0} \geq 0$. By Lemma 20, E_0 is closed and $G_0 = F$ on E_0, and G_0 is affine on every pool, i.e., on every component of $(\underline{\theta}, c) \setminus E_0$.

Finitely many pools. Let (a, b) be a pool with $a, b \in E_0$. Then G_0 is the chord of F on $[a, b]$, and $\int_a^b (F - G_0)\, dt = \Delta(b) - \Delta(a) = 0$. If (a, b) contained no z_j, then F would be strictly convex or strictly concave on $[a, b]$, and $F - G_0$ would have a strict sign on (a, b), which is impossible. So each such pool contains some z_j with $1 \leq j \leq m - 1$, and, as pools are disjoint, there are at most $m - 1$ of them. Every other pool has an endpoint outside E_0, which can only be its upper end $c < \overline{\theta}$. So there are at most m pools.

(H2). So $[\underline{\theta}, c]$ is the union of the closures of the pools and the components of E_0, finitely many intervals in all. Each pool closure is a block: G_0 is affine on it, and it meets E_0 only at its ends. A degenerate component of E_0 lies in a block. On a nondegenerate component J, $F = G_0$ is convex, so f is nondecreasing on J and, being strictly monotone on each $[z_{j-1}, z_j]$, strictly increasing. So F is strictly convex on J and twice continuously differentiable on its interior except at the points z_j, and J is a revealed stretch.

(H3). Let $x_0 \in E_0$ be the lower end of a pool on which G_0 has slope γ. Take $r > 0$ with $(x_0, x_0 + r)$ inside the pool and inside one $[z_{j-1}, z_j]$, and such that $f - \gamma$ has a

constant strict sign on it: by continuity if $f(x_0) \neq \gamma$, and by strict monotonicity if $f(x_0) = \gamma$. As $F(x_0) = G_0(x_0)$ and $\Delta(x_0) = 0$,

$$\Delta(x) = \int_{x_0}^{x} \int_{x_0}^{t} (f - \gamma)\, ds\, dt, \qquad x \in (x_0, x_0 + r),$$

which would be negative if $f < \gamma$ there. So $f > \gamma$ on $(x_0, x_0 + r)$. At an upper end, $\Delta(x) = \int_x^{x_0} \int_t^{x_0} (f - \gamma)\, ds\, dt$ for $x \in (x_0 - r, x_0)$, and likewise $f > \gamma$ there.

□

C.8.13 Proof of Theorem 4

Under the assumption of Theorem 4, the noiseless equilibrium is well behaved (Proposition 14). I prove the theorem for every F with a well-behaved noiseless equilibrium.

Lemma 22. *Let F be a prior, $G \leq_{cx} F$ with $\underline{\theta} \in \operatorname{supp}(G)$, and $\psi : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ convex, continuous and nondecreasing, with $\psi(\underline{\theta}) \geq 0$, $\psi \leq 1$ on $\operatorname{supp}(G)$, $\int \psi\, dG = \frac{1}{2}$, and $d\psi'$ carried by $\{I_G = I_F\}$. Suppose that, for every t,*

$$\int_{[t,\overline{\theta}]} \psi\, dG \leq \frac{1 - G(t^-)^2}{2} \qquad \text{and} \qquad \int_{(t,\overline{\theta}]} \psi\, dG \leq \frac{1 - G(t)^2}{2}.$$

Then some general selection rule nondecreasing in the own posterior mean has the equilibrium (G, G), the agent's value function equals ψ on $\operatorname{supp}(G)$, and the firm's utility is $2 \int s\, \psi(s)\, dG(s)$.

Proof. Let $Y := \psi(s)$ for $s \sim G$; it has mean $\frac{1}{2}$ and values in $[0, 1]$. As ψ is nondecreasing, its upper level sets are intervals $[t, \overline{\theta}]$ or $(t, \overline{\theta}]$, so the two inequalities are the bound of the necessity part of the proof of Lemma 15 at every upper level set, and the interpolation argument there gives $Y \leq_{cx} U[0, 1]$; then Lemmas 15 and 17(i), applied

with G in place of F, give a rule nondecreasing in the own posterior mean, a lottery over coarse gradings, with $V_G = \psi$ on $\operatorname{supp}(G)$. A posterior mean x off the support has the same quantile as the highest support point $r(x)$ below it, which exists as $\underline{\theta} \in \operatorname{supp}(G)$, so it is graded like $r(x)$, and $V_G(x) = \psi(r(x)) \leq \psi(x)$, as ψ is nondecreasing. For every $G' \leq_{cx} F$, by (1), $\int V_G \, dG' \leq \int \psi \, dG' \leq \int \psi \, dF = \int \psi \, dG = \frac{1}{2} = \int V_G \, dG$. So (G, G) is an equilibrium, and Lemma 14 gives $u_F = p + 2\operatorname{Cov}_G(s, \psi(s)) = 2\int s\,\psi \, dG$. □

From now on, let F have a well-behaved noiseless equilibrium (G_0, G_0) with firm's utility u_0, and let c, E_0 and the certificate ψ_0 be as in Lemma 20; let g_0 be the right derivative of G_0 and $\Delta := I_F - I_{G_0} \geq 0$. A *pool* is a component of $(\underline{\theta}, c) \setminus E_0$. By (H2), merging adjacent blocks with equal slopes, $[\underline{\theta}, c]$ is a finite union of maximal revealed stretches and maximal blocks. Two adjacent maximal blocks meet at a point of E_0 where g_0 jumps up, a *kink*, as G_0 is convex and dg_0 is carried by E_0. A *chain* is a maximal interval $[z_0, z_1]$ that is a union of adjacent maximal blocks $B_i = [b_{i-1}, b_i]$, $i = 1, \ldots, K$, with $b_0 = z_0$, $b_K = z_1$, slopes $\gamma_1 < \cdots < \gamma_K$ and widths $W_i := b_i - b_{i-1}$. Then $z_0 = \underline{\theta}$ or z_0 is the upper end of a revealed stretch, and $z_1 = c$ or z_1 is the lower end of one.

Proposition 15. *Let F have a well-behaved noiseless equilibrium (G_0, G_0), and let F not be convex. Let u_0 be the firm's utility in (G_0, G_0), $c := \max \operatorname{supp}(G_0)$, $\psi_0 := G_0$ on $[\underline{\theta}, c]$ and $\psi_0(x) := 1 + G_0'(c^-)(x - c)$ on $[c, \overline{\theta}]$, and let P and S be the unions of the components of $(\underline{\theta}, c) \setminus \{I_{G_0} = I_F\}$ and of the revealed stretches. For every $\vartheta \in [0, \frac{1}{2})$, there are a continuous, piecewise twice continuously differentiable h on $[\underline{\theta}, \overline{\theta}]$ and $\varepsilon_0 > 0$ such that, with $H(t) := \int_{[\underline{\theta}, t]} h \, dG_0$,*

$$\Phi_\vartheta := (2 - 4\vartheta) \int_P H(t)\, dt - 2 \int_S H(t)\, dt > 0,$$

and, for every $\varepsilon \in (0, \varepsilon_0)$, some general selection rule nondecreasing in the own posterior mean has a symmetric equilibrium $(G_\varepsilon, G_\varepsilon)$ in which the agent's value function equals $\psi_0 + \varepsilon h$ on

$\operatorname{supp}(G_\varepsilon)$ *and the firm's utility* $u(\varepsilon)$ *satisfies*

$$\varepsilon\,\Phi_\vartheta - o(\varepsilon) \le u(\varepsilon) - u_0 \le \varepsilon\,\Phi_\vartheta.$$

The equilibrium signal is built from the following tangency lemma, which is applied at every contact point of a chain.

Lemma 23. *Let* $r > 0$*, and let* A *and* Γ *be continuously differentiable and convex on* $[a, a+r]$*, with* A' *strictly increasing,* $\Gamma'' = \gamma > 0$*,* $A(a) = \Gamma(a)$*,* $A'(a) = \Gamma'(a)$*,* $A > \Gamma$ *on* $(a, a+r]$ *and* $A' - \Gamma'$ *nondecreasing. Let* $T(x) = T_0 + T_1(x-a) + \frac{\tau}{2}(x-a)^2$ *with* $\tau \le 0$*,* $T \ge 0$ *on* $[a, a+r]$*, and either* $T_0 > 0$ *or* $T_0 = 0 < T_1$*, and let* $\Lambda := \Gamma + \varepsilon T$*. For all sufficiently small* $\varepsilon > 0$*, there are* $a < x_1 < x_2 < a + r$ *such that*

(i) the tangent ℓ *of* A *at* x_1 *is the tangent of* Λ *at* x_2*;*

(ii) $\Lambda \le A$ *on* $[x_2, a+r]$*;*

(iii) $\Gamma \le \ell \le \Lambda$ *on* $[x_1, x_2]$ *and* $A \le \Lambda$ *on* $[a, x_1]$*;*

(iv) $x_2 \to a$ *as* $\varepsilon \to 0$*.*

Proof. Let $\varepsilon < \gamma/(2|\tau|)$, so that $\Lambda'' \ge \frac{\gamma}{2}$, and let $\rho := \varepsilon T - (A - \Gamma)$. As $\rho' = \varepsilon T' - (A' - \Gamma')$ is nonincreasing, ρ is concave; $\rho(a) = \varepsilon T_0 \ge 0$, $\rho > 0$ just right of a (if $T_0 = 0$, as $\rho'(a) = \varepsilon T_1 > 0$), and $\rho(a+r) < 0$ for small ε. So ρ has a unique zero $x^* \in (a, a+r)$, $\rho > 0$ on (a, x^*), $\rho < 0$ on $(x^*, a+r]$ and $\rho'(x^*) < 0$; and $x^* \to a$ as $\varepsilon \to 0$, as $A - \Gamma$ is bounded away from zero on $[a+\eta, a+r]$ for every $\eta > 0$. By $\rho'(x^*) < 0$, $A'(x^*) > \Lambda'(x^*) \ge \Lambda'(a)$, and $A'(x^*) > A'(a)$. For $\sigma \in [\sigma_0, A'(x^*)]$ with $\sigma_0 := \max\{A'(a), \Lambda'(a)\}$, let $x_1(\sigma) \in [a, x^*]$ and, for small ε, $x_2(\sigma) \in [a, a+r]$ solve $A'(x_1) = \sigma$ and $\Lambda'(x_2) = \sigma$, and let $\tau(\sigma)$ be the intercept of the tangent of A at x_1 minus that of the tangent of Λ at x_2; it is continuous. At σ_0, $\tau < 0$: if $\sigma_0 = \Lambda'(a)$, then $x_2 = a$, and if moreover $x_1 > a$, the tangent of A at x_1

lies strictly below $A(a) = \Gamma(a) \leq \Lambda(a)$ at a, while if $x_1 = a$, then $T_0 > 0$, as $T_0 = 0$ gives $\Lambda'(a) > A'(a)$, and $\tau = -\varepsilon T_0$; if $\sigma_0 = A'(a) > \Lambda'(a)$, then $x_1 = a$, $x_2 > a$, the tangent of A at a is that of Γ, and $\tau \leq \Gamma(x_2) - \Lambda(x_2) = -\varepsilon T(x_2) < 0$, as $T > 0$ on $(a, a + r)$ near a. At $A'(x^*)$, $x_1 = x^*$ and $\tau \geq A(x^*) - \Lambda(x^*) = -\rho(x^*) = 0$, as the tangent of the convex Λ lies below it. Take a zero of τ: the two tangents coincide, ℓ. As $\ell \leq A$, $\rho(x_2) = \ell(x_2) - A(x_2) \leq 0$, and $x_2 \neq a$ as $\tau(\sigma_0) < 0$, so $x_2 \geq x^* \geq x_1$, and $x_1 = x_2$ would give $A'(x^*) = \Lambda'(x^*)$; so $x_1 < x_2$, which proves (i). (ii) holds as $\rho \leq 0$ on $[x^*, a + r]$. In (iii), $\ell \leq \Lambda$ by convexity, $\ell \geq \Gamma$ at x_1 and x_2 and hence between them, as Γ is convex, and $\rho \geq 0$ on $[a, x_1]$. For (iv), $\frac{\gamma}{2}(x_2 - a) \leq \Lambda'(x_2) - \Lambda'(a) \leq A'(x^*) - A'(a) + \varepsilon|T_1| \to 0$. □

By the reflection $x \mapsto 2b - x$, which preserves all hypotheses, the same holds at an upper endpoint b, on $[b - r, b]$, with either $T(b) > 0$ or $T(b) = 0 > T'(b)$, and $b - r < x_2 < x_1 < b$.

Proof of Proposition 15. As F is not convex, G_0 has a block: otherwise $[\underline{\theta}, c] \subseteq E_0$, so $G_0 = F$ on $[\underline{\theta}, c]$, and $c = \overline{\theta}$, as $c \notin E_0$ if $c < \overline{\theta}$; then $F = G_0$ would be convex. So a chain $[z_0, z_1]$ exists; fix one and $\vartheta \in [0, \frac{1}{2})$.

Step 1 (the direction on the chain). On each block B_i of the chain, let H be the following polynomial of degree at most two and $h := H'/\gamma_i$:

(a) if $K = 1$, $z_0 = \underline{\theta}$ and $z_1 = c$: $H(t) = 4(t - \underline{\theta})(c - t)/W_1^2$;

(b) if $K = 1$, $z_0 > \underline{\theta}$ and $z_1 = c$: $H(t) = 1 - (t - z_0)/W_1$;

(c) otherwise: $H(t) = 1 - \big((b_1 - t)/W_1\big)^2$ on B_1 if $z_0 = \underline{\theta}$, $H(t) = 1 - \big((t - b_{K-1})/W_K\big)^2$ on B_K if $z_1 = c$, and $H \equiv 1$ on every other block.

If $z_1 = c$, extend h affinely to $[b_{K-1}, \overline{\theta}]$. Then h is affine and nonincreasing on each block, $H \geq 0$ on $[z_0, z_1]$, and $H > 0$ on the interior of each block. At each kink, $H = 1$ and $h = 0$ on both sides; $h(\underline{\theta}) > 0$ if $z_0 = \underline{\theta}$, and $h(c) < 0$ if $z_1 = c$. If $z_1 < c$, then

$h(z_1) = 0 \geq h'(z_1^-)$. If $z_0 > \underline{\theta}$, then $h'(z_0^+) = 0$, and $h(z_0) = 0$ except in case (b), where $h(z_0) = -1/(\gamma_1 W_1)$.

Step 2 (sinks). Let $\delta > 0$. If $z_0 > \underline{\theta}$, let $[s, z_0]$ be the revealed stretch below z_0, take $s' \in (s, z_0)$ such that F is twice continuously differentiable on (s', z_0), which (H2) allows, let $\bar{f} := \max_{[s', z_0]} f$, and assume $\delta < z_0 - s'$. As f' is continuous on (s', z_0), nonnegative, as F is convex, and vanishes on no interval, as F is strictly convex, there is a window $[w, w'] \subseteq (z_0 - \delta, z_0)$ on which $f' \geq m$ for some $m > 0$. Outside case (b), let $h := \lambda(t - w)^2(w' - t)^2$ on $[w, w']$ and $h := 0$ elsewhere on $[s, z_0]$, with $\lambda > 0$ such that $\int_w^{w'} h \, dF = 1$; then H rises from 0 to 1 on $[w, w']$ and equals 1 on $[w', z_0]$. In case (b), with $h_{z_0} := -1/(\gamma_1 W_1)$, take the window $[w', w]$ with $w - w' \leq z_0 - w$, let $h := 0$ on $[s, w']$, $h(t) := h_{z_0} + k(t - z_0)$ on $[w, z_0]$, and let h on $[w', w]$ be the cubic Hermite interpolant from value 0 and slope 0 at w' to value $h_{z_0} + k(w - z_0)$ and slope k at w. Then $\int_{w'}^{u} h \, dF$ is affine and strictly decreasing in k, with slope at most $2\bar{f}\,(z_0 - w')^2$ in absolute value, and nonpositive at $k = 0$. So k with $\int_{w'}^{u} h \, dF = 1$ is negative and $h(w) = h_{z_0} + |k|(z_0 - w) \geq h_{z_0} + \frac{1}{4\bar{f}\,(z_0 - w')} > 0$ for small δ. Then $h \geq 0$ on $[w', w]$, as both Hermite terms are nonnegative, and h changes sign once on $[w, z_0]$, so H rises from 0 and then falls to $H(z_0) = 1$, and $h'(z_0^-) = k < 0 = h'(z_0^+)$. If $z_1 < c$, let $[z_1, e]$ be the revealed stretch above z_1, take $e' \in (z_1, e)$ such that F is twice continuously differentiable on (z_1, e'), assume $\delta < e' - z_1$, and take a window $[w, w'] \subseteq (z_1, z_1 + \delta)$ with $f' \geq m$, and let $h := -\mu(t - w)^2(w' - t)^2$ there and $h := 0$ elsewhere on $[z_1, e]$, with $\int_w^{w'} h \, dF = -1$; then $h(z_1) = h'(z_1^+) = 0$ and H falls from 1 to 0. Let $h := 0$ everywhere else. Then $H \geq 0$, H vanishes outside the chain and the at most two revealed stretches next to it, $H(c) = 0$, and $\int_S H \leq 2\delta(1 + |h_{z_0}|\bar{f}\,\delta)$. As $\int_P H > 0$ does not depend on δ, $\Phi_\vartheta > 0$ for small δ; fix such a δ.

Step 3 (the certificate). Let $\psi_\varepsilon := \psi_0 + \varepsilon h$. On a revealed stretch, h vanishes except on $[z_0 - \delta, z_0] \subseteq (s', z_0]$ and $[z_1, z_1 + \delta] \subseteq [z_1, e')$, so $\psi_\varepsilon = F$ is convex off

these sets. On (s', z_0) and (z_1, e'), F is twice continuously differentiable with $f' \geq 0$, and h is continuously differentiable, with $h'' \neq 0$ only on the windows, where $\psi_\varepsilon'' \geq m - \varepsilon \sup |h''| \geq 0$ for small ε; in case (b), h is affine on $[w, z_0]$. So ψ_ε is convex on (s', z_0) and (z_1, e'), and hence, as convexity is local and ψ_ε is continuous, on each revealed stretch. On blocks and above c, ψ_ε is affine, also across contact points inside a block, where h has no kink. At a kink b_i, ψ_0' jumps up by $\gamma_{i+1} - \gamma_i > 0$, which dominates the jump of $\varepsilon h'$ for small ε; at z_0 and z_1, $h'(z_0^-) \leq h'(z_0^+)$ and $h'(z_1^-) \leq h'(z_1^+)$, and ψ_0 is convex. So ψ_ε is convex and continuous. It is nondecreasing for small ε, as $h' \neq 0$ only where ψ_0' is bounded away from zero: on blocks and above c, where $\psi_0' \geq \gamma_1$, and on compact subsets of $(s', z_0]$ and (z_1, e'), where $\psi_0' = f$ is continuous and positive, as f is nondecreasing on a revealed stretch and F is strictly increasing; and $\psi_\varepsilon(\underline{\theta}) = \varepsilon h(\underline{\theta}) \geq 0$. As $\psi_\varepsilon(c) = 1 + \varepsilon h(c) \leq 1$, $\psi_\varepsilon \leq 1$ on $[\underline{\theta}, c]$. Finally, $d\psi_\varepsilon'$ is carried by E_0, on which $d\psi_0'$ is carried, together with the sink windows and the kinks of h, all of which lie in $S \cup E_0$.

Step 4 (the signal). On each pool Q of the chain, with slope γ, let $T := (1 - \vartheta)H/\gamma$, a polynomial of degree at most two with $T' = (1 - \vartheta)h$ nonincreasing, $T \geq 0$, and $T > 0$ inside Q; let $\Lambda_Q := I_{G_0} + \varepsilon T$. At each endpoint $x_0 \in E_0$ of Q, apply Lemma 23, or its reflection, with $A := I_F$ and $\Gamma := I_{G_0}$, on a one-sided neighbourhood of x_0 in Q of length r on which $\Delta > 0$ by (H2) and $f \geq \gamma$ almost everywhere by (H3), so that $A' - \Gamma' = F - G_0$ is nondecreasing; $A' = F$ is strictly increasing, as $f \geq \gamma > 0$ almost everywhere there. Its conditions on T hold: $T(x_0) > 0$ at every contact point in $(\underline{\theta}, \overline{\theta})$, as $H(x_0) > 0$ there; $T(\underline{\theta}) = 0 < T'(\underline{\theta})$ if $x_0 = \underline{\theta}$; and $T(\overline{\theta}) = 0 > T'(\overline{\theta})$ if $x_0 = \overline{\theta} = c$. The *revealed part* of the resulting layer is the interval between x_0 and x_1, and its *gap* is

(x_1, x_2), or (x_2, x_1). Define K by

$$K := \begin{cases} I_F & \text{on revealed parts,} \\ \ell & \text{on gaps,} \\ \Lambda_Q & \text{on the rest of each pool } Q \text{ of the chain,} \\ I_{G_0} & \text{elsewhere.} \end{cases}$$

For small ε, $\Lambda_Q \le I_F$ on the rest of Q: near the contact points by Lemma 23(ii), and elsewhere, also near c when $c < \overline{\theta}$, as Δ is bounded away from zero there while $\varepsilon T \le \varepsilon \sup T$. The pieces of K are convex and meet with equal values and slopes, except at c when $z_1 = c < \overline{\theta}$, where $K'(c^-) = 1 + \varepsilon(1-\vartheta)h(c) < 1 = K'(c^+)$. So K is convex, $K \le I_F$, $K(\overline{\theta}) = I_F(\overline{\theta})$, and $G_\varepsilon := K'$ is a CDF with $G_\varepsilon \le_{cx} F$, $\operatorname{supp}(G_\varepsilon) \subseteq [\underline{\theta}, c]$ and $\underline{\theta} \in \operatorname{supp}(G_\varepsilon)$. It is continuous except for an atom of mass $\alpha := -\varepsilon(1-\vartheta)h(c)$ at c if $z_1 = c < \overline{\theta}$; let $\alpha := 0$ otherwise. Let $J := (K - I_{G_0})/\varepsilon$ and $D := J' = (G_\varepsilon - G_0)/\varepsilon$, which is continuous on $[\underline{\theta}, c)$. Then $J = 0$ outside the pools of the chain and on E_0; on these pools, $0 \le J \le T$ by Lemma 23(iii), and $J = T$ outside the layers, whose total length $\lambda(\varepsilon)$ tends to zero by Lemma 23(iv). As $K = I_F$ on $S \cup E_0$, $d\psi'_\varepsilon$ is carried by $\{K = I_F\}$.

Step 5 (totals). As $J = 0$ on E_0 and $dh' = 0$ on the pools, (1) gives $\int \psi_0 \, dG_\varepsilon = \frac{1}{2} + \varepsilon \int J \, d\psi'_0 = \frac{1}{2}$ and $\int h \, dG_\varepsilon = \int h \, dG_0 + \varepsilon \int J \, dh' = H(c) = 0$. So $\int \psi_\varepsilon \, dG_\varepsilon = \frac{1}{2}$.

Step 6 (the rival's scores). Let $B(t) := \int_{[t,\overline{\theta}]} \psi_\varepsilon \, dG_\varepsilon - \frac{1-G_\varepsilon(t^-)^2}{2}$. For $t > c$, $B(t) = 0$; $B(c) = \alpha\left(\varepsilon h(c) + \frac{\alpha}{2}\right) = \frac{1+\vartheta}{2}\alpha\varepsilon h(c) \le 0$, and with the atom excluded the value is 0. For $t < c$, as G_ε is continuous on $[t, c)$ and $\psi_\varepsilon - G_\varepsilon = \varepsilon(h - D)$ there,

$$B(t) = B(c) + \int_{[t,c)} (\psi_\varepsilon - G_\varepsilon) \, dG_\varepsilon = B(c) + \varepsilon \int_t^c (h - D) \, dG_0 + \varepsilon^2 \int_{[t,c)} (h - D) \, dD.$$

Integrating by parts, with $H(c) = 0$, $J(c) = 0$, $D(c^-) = -\alpha/\varepsilon$, and $\int_t^c J \, dg_0 = \int_t^c J \, dh' = 0$,

gives

$$B(t) = -\varepsilon\big(H - g_0 J\big)(t) + \varepsilon^2\Big[D\Big(\frac{D}{2} - h\Big) + h'J\Big](t),$$

as the terms at c^- cancel $B(c)$. (i) On the rest of a pool, $J = T$, $g_0 = \gamma$ and $D = (1-\vartheta)h$, so $B = -\varepsilon\vartheta H - \varepsilon^2\frac{1-\vartheta^2}{2}h^2 + \varepsilon^2 h'T \le 0$. (ii) On a gap, G_ε is constant, so B is constant and equals its value at the end x_2, which is ≤ 0 by (i). (iii) On a maximal interval R within the closure of the pools of the chain on which $G_\varepsilon = F$, which contains at most one contact point, $B' = -f(\psi_\varepsilon - F) = f\varphi$ with $\varphi := F - G_0 - \varepsilon h$, which is continuous and nondecreasing, as $\varphi' = f - \gamma - \varepsilon h' \ge 0$ almost everywhere on each side of the contact point by (H3) and $h' \le 0$. So B is nonincreasing and then nondecreasing on R, and is at most its value at an end of R: a gap end, a point of a revealed stretch, $\underline{\theta}$, where $B = \int \psi_\varepsilon\, dG_\varepsilon - \frac{1}{2} = 0$, or $\overline{\theta}$, where $B = 0$. (iv) Elsewhere, $J = D = 0$, so $B = -\varepsilon H \le 0$. Hence both inequalities of Lemma 22 hold, as they differ only at the atom. With Steps 3–5, Lemma 22 gives a rule nondecreasing in the own posterior mean with the equilibrium $(G_\varepsilon, G_\varepsilon)$ and value function ψ_ε on $[\underline{\theta}, c]$.

Step 7 (payoff). By Lemma 22, $u(\varepsilon) - u_0 = 2\int s\psi_\varepsilon\, dG_\varepsilon - 2\int s\psi_0\, dG_0$. By (1), with $d(s\psi_0)' = 2\psi_0'\, ds + s\, d\psi_0'$, $d(sh)' = 2h'\, ds + s\, dh'$, $J = 0$ on E_0 and above c, and $\int s\, h\, dG_0 = -\int H\, ds$,

$$u(\varepsilon) - u_0 = \varepsilon\Big[4\int g_0 J\, ds - 2\int H\, ds\Big] + 4\varepsilon^2\int Jh'\, ds.$$

As $J \le T = (1-\vartheta)H/\gamma$ on the pools of the chain, $J = 0$ off them, $H = 0$ outside $P \cup S$, and $Jh' \le 0$, $u(\varepsilon) - u_0 \le \varepsilon\Phi_\vartheta$. As $J = T$ off the layers, $u(\varepsilon) - u_0 \ge \varepsilon\Phi_\vartheta - 4\varepsilon\gamma_K \sup T\, \lambda(\varepsilon) - 4\varepsilon^2 \sup T \sup|h'|(c - \underline{\theta}) = \varepsilon\Phi_\vartheta - o(\varepsilon)$. □

Proof of Theorem 4. By Lemma 20 and interchangeability, (G_0, G_0) is the only equilibrium of the noiseless rule. If F is convex, $G_0 = F$, and the noiseless rule attains the first best by Proposition 10; no rule exceeds it, as $u_F \le \mathbb{E}[\max(s_1, s_2)] \le \mathbb{E}[\max(\theta_1, \theta_2)]$

in every equilibrium. This direction holds for every prior, by Proposition 10. If F is not convex, Proposition 15, with any $\vartheta \in [0, \frac{1}{2})$, gives $u(\varepsilon) > u_0$ for small $\varepsilon > 0$. □